Nested Bethe Ansatz for RTT–Algebra $\mathcal{A}_n$

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Abstract

This paper continues our recent studies on the algebraic Bethe ansatz for the RTT–algebras of sp$(2n)$ and o$(2n)$ types. In these studies, we encountered the RTT–algebras which we called $\mathcal{A}_n$. The next step in our construction of the Bethe vectors for the RTT-algebras of type sp$(2n)$ and o$(2n)$ is to find the Bethe vectors for the RTT–algebras $\mathcal{A}_n$. This paper deals with the construction of the Bethe vectors of the RTT–algebra $\mathcal{A}_n$ using the Bethe vectors of the RTT–algebra $\mathcal{A}_{n-1}$.

1 Introduction

In studying the algebraic Bethe ansatz for the RTT–algebras of type sp$(2n)$ and o$(2n)$ [1, 2], we discovered some the RTT–algebras which we called $\mathcal{A}_n$. The main result of these works is the assertion that for the construction of eigenvalues and eigenvectors of the transfer–matrix of the RTT–algebras of type sp$(2n)$ and o$(2n)$ it is enough to find eigenvalues and eigenvectors for the RTT–algebra $\mathcal{A}_n$.

In this work, we deal with the nested Bethe ansatz for the RTT–algebra $\mathcal{A}_n$. We show how to construct eigenvectors for the RTT–algebra $\mathcal{A}_n$ by using eigenvectors of the RTT–algebra $\mathcal{A}_{n-1}$.

Note, that the RTT–algebra $\mathcal{A}_{n-1}$ is not the RTT–subalgebra $\mathcal{A}_n$. However, $\mathcal{A}_n$ contains two the RTT–subalgebras $\mathcal{A}_n^{(+)}$ and $\mathcal{A}_n^{(-)}$, which are of type gl(n). The RTT–algebras $\mathcal{A}_{n-1}^{(\pm)}$ are already the RTT–subalgebras of $\mathcal{A}_n^{(\pm)}$. As we will see later, we can construct some eigenvectors for the RTT–algebras $\mathcal{A}_n$ as Bethe vectors of the RTT–algebras $\mathcal{A}_n^{(\pm)}$, i.e. as the Bethe vectors for the RTT–algebras of the type gl(n). Our result for such eigenvectors is the same as for the nested Bethe ansatz for the RTT–algebras of gl(n), which can be found in [3]. In this sense, our construction is a certain generalization of the nested Bethe ansatz for the RTT–algebras of type gl(n).

The proofs of many claims are only a suitable, but long adjustment of the Yang–Baxter and the RTT–equations. We have included them in Appendix for better clarity of the main text.

2 The RTT–algebra $\mathcal{A}_n$

We denote $\mathbf{E}^i_k$ and $\mathbf{E}^{-i}_{-k}$, where $i,k=1,\ldots,n$, the matrices $\left(\mathbf{E}^i_k\right)^r_s=\left(\mathbf{E}^{-i}_{-k}\right)^{-r}_{-s}=\delta^r_k\delta^i_s$. Then the relations $\mathbf{E}^i_k\mathbf{E}^r_s=\delta^i_s\mathbf{E}^r_k$, $\sum\limits_{i=1}^n\mathbf{E}^i_i=\mathbf{I}_+$ and $\sum\limits_{i=1}^n\mathbf{E}^{-i}_{-i}=\mathbf{I}_-$ apply.

The RTT–algebra $\mathcal{A}_n$ is an associative algebra with a unit that is generated by the elements $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where $i,k=1,\ldots,n$. If we introduce the monodromy matrix $\mathbf{T}(x)=\mathbf{T}^{(+)}(x)+\mathbf{T}^{(-)}(x)$, where

$$\mathbf{T}^{(+)}(x)=\sum_{i,k=1}^{n}\mathbf{E}^k_i\otimes T^i_k(x)\,,\qquad \mathbf{T}^{(-)}(x)=\sum_{i,k=1}^{n}\mathbf{E}^{-k}_{-i}\otimes T^{-i}_{-k}(x)\,,$$

the commutation relations between generators are defined by the RTT–equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{T}_1(x)\mathbf{T}_2(y)=\mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(x,y), \tag{1}$$

where R–matrix is $\mathbf{R}(x,y)=\mathbf{R}^{(+,+)}(x,y)+\mathbf{R}^{(+,-)}(x,y)+\mathbf{R}^{(-,+)}(x,y)+\mathbf{R}^{(-,-)}(x,y)$,

$$\begin{aligned}
\mathbf{R}^{(+,+)}(x,y)&=\frac{1}{f(x,y)}\Big(\mathbf{I}_+\otimes\mathbf{I}_+ + g(x,y)\sum_{i,k=1}^{n}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\Big),\\
\mathbf{R}^{(+,-)}(x,y)&=\mathbf{I}_+\otimes\mathbf{I}_- - k(x,y)\sum_{i,k=1}^{n}\mathbf{E}^i_k\otimes\mathbf{E}^{-i}_{-k}\,,\\
\mathbf{R}^{(-,+)}(x,y)&=\mathbf{I}_-\otimes\mathbf{I}_+ - h(x,y)\sum_{i,k=1}^{n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\,,\\
\mathbf{R}^{(-,-)}(x,y)&=\frac{1}{f(x,y)}\Big(\mathbf{I}_-\otimes\mathbf{I}_- + g(x,y)\sum_{i,k=1}^{n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{-k}_{-i}\Big),\\
g(x,y)&=\frac{1}{x-y}\,,\qquad f(x,y)=\frac{x-y+1}{x-y}\,,\\
h(x,y)&=\frac{1}{x-y+n-\eta}\,,\qquad k(x,y)=\frac{1}{x-y+\eta}
\end{aligned}$$

and η is any number. For $\eta=-1$ we obtain the RTT–algebra connected with the RTT–algebra of sp$(2n)$ type and for $\eta=1$ the RTT–algebra connected with the RTT-algebra of o$(2n)$ type.
By direct calculation, it can be verified that this R–matrix satisfies the Yang–Baxter equation

$$\mathbf{R}_{1,2}(x,y)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{2,3}(y,z)=\mathbf{R}_{2,3}(y,z)\mathbf{R}_{1,3}(x,z)\mathbf{R}_{1,2}(x,y) \tag{2}$$

and has the inverse R–matrix

$$\big(\mathbf{R}(x,y)\big)^{-1}=\big(\mathbf{R}^{(+,+)}(x,y)\big)^{-1}+\big(\mathbf{R}^{(+,-)}(x,y)\big)^{-1}+\big(\mathbf{R}^{(-,+)}(x,y)\big)^{-1}+\big(\mathbf{R}^{(-,-)}(x,y)\big)^{-1}$$

where

$$\begin{aligned}
\big(\mathbf{R}^{(+,+)}(x,y)\big)^{-1}&=\frac{1}{f(y,x)}\Big(\mathbf{I}_+\otimes\mathbf{I}_+ + g(y,x)\sum_{i,k=1}^{n}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\Big),\\
\big(\mathbf{R}^{(+,-)}(x,y)\big)^{-1}&=\mathbf{I}_+\otimes\mathbf{I}_- - h(y,x)\sum_{i,k=1}^{n}\mathbf{E}^i_k\otimes\mathbf{E}^{-i}_{-k}\,,\\
\big(\mathbf{R}^{(-,+)}(x,y)\big)^{-1}&=\mathbf{I}_-\otimes\mathbf{I}_+ - k(y,x)\sum_{i,k=1}^{n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\,,\\
\big(\mathbf{R}^{(-,-)}(x,y)\big)^{-1}&=\frac{1}{f(y,x)}\Big(\mathbf{I}_-\otimes\mathbf{I}_- + g(y,x)\sum_{i,k=1}^{n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^{-k}_{-i}\Big)\,.
\end{aligned}$$

Therefore, it defines the RTT–algebra that we denote by $\mathcal{A}_n$.
The explicit form of commutation relations between generators of the RTT–algebra $\mathcal{A}_n$ is given in the Appendix.

It is easily seen that the RTT–equation (1) can be written as

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(x,y)\,, \tag{3}$$

where ϵ_1, $\epsilon_2 = \pm$. From this form of the RTT–equation it is clear that in the RTT–algebra $\mathcal{A}_n$ there are two RTT-subalgebras $\mathcal{A}_n^{(+)}$ and $\mathcal{A}_n^{(-)}$, which are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\, k = 1,\, \ldots,\, n$.
Using the RTT–equation (3), it is possible to show that in the RTT–algebra $\mathcal{A}_n$ the operators

$$H^{(+)}(x) = \mathrm{Tr}\,\mathbf{T}^{(+)}(x) = \sum_{i=1}^{n} T_i^i(x)\,, \qquad H^{(-)}(x) = \mathrm{Tr}\,\mathbf{T}^{(-)}(x) = \sum_{i=1}^{n} T_{-i}^{-i}(x)$$

mutually commute.
We deal with the representations of the RTT–algebra $\mathcal{A}_n$ on the vector space $\mathcal{W} = \mathcal{A}_n\omega$, where ω is a vacuum vector for which the relations

$$\begin{array}{llll} T_k^i(x)\omega = 0 & \text{for} \quad 1 \le i < k \le n\,, & & T_i^i(x)\omega = \lambda_i(x)\,,\omega \\ T_{-i}^{-k}(x)\omega = 0 & \text{for} \quad 1 \le i < k \le n\,, & & T_{-i}^{-i}(x)\omega = \lambda_{-i}(x)\omega \end{array}$$

hold. Our goal is to find in the vector space $\mathcal{W}$ common eigenvectors of the operators $H^{(\pm)}(x)$.

In the RTT–algebra $\mathcal{A}_n$ there are two RTT–subalgebras $\tilde{\mathcal{A}}^{(+)} = \mathcal{A}_{n-1}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)} = \mathcal{A}_{n-1}^{(-)}$ of $\mathrm{gl}(n-1)$ type, which are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\, k = 1,\, \ldots, n-1$.
First, we will consider the subspace $\tilde{\mathcal{W}}$ generated by the elements $\tilde{\mathcal{A}}^{(+)}\tilde{\mathcal{A}}^{(-)}\omega$.

Proposition 1. The relations

$$T_n^i(x)w = T_{-i}^{-n}(x)w = 0\,, \qquad T_n^n(x)w = \lambda_n(x)w\,, \qquad T_{-n}^{-n}(x)w = \lambda_{-n}(x)w \tag{4}$$

hold for any $w \in \tilde{\mathcal{W}}$ and $i = 1,\, 2,\, \ldots,\, n-1$.

Proof. First, we consider the space $\tilde{\mathcal{W}}^{(-)} = \tilde{\mathcal{A}}^{(-)}\omega \subset \tilde{\mathcal{W}}$. To prove relation (4) for $w = w^{(-)} \in \mathcal{W}^{(-)}$, it is sufficient to show that if (4) is valid for $w^{(-)}$, it also applies to $T_{-s}^{-r}(y)w^{(-)}$, where $r, s = 1, \ldots, n-1$. From the commutation relations we get for $i = 1,\, \ldots,\, n$ and $r,\, s = 1,\, \ldots,\, n-1$

$$T_{-i}^{-n}(x)T_{-s}^{-r}(y) = T_{-s}^{-r}(y)T_{-i}^{-n}(x) + g(x,y)T_{-i}^{-r}(y)T_{-s}^{-n}(x) - g(x,y)T_{-i}^{-r}(x)T_{-s}^{-n}(y)$$

It follows that for any $w^{(-)} \in \tilde{\mathcal{W}}^{(-)}$ we have

$$T_{-i}^{-n}(x)w^{(-)} = 0 \quad \text{for} \quad i = 1,\, \ldots,\, n-1\,, \qquad T_{-n}^{-n}(x)w^{(-)} = \lambda_{-n}(x)w^{(-)}\,.$$

For any $r,\, s = 1,\, \ldots,\, n-1$, the commutation relations give

$$T_n^n(x)T_{-s}^{-r}(y) = T_{-s}^{-r}(y)T_n^n(x)\,,$$

which proves that $T_n^n(x)w^{(-)} = \lambda_n(x)w^{(-)}$ for any $w^{(-)} \in \tilde{\mathcal{W}}^{(-)}$.

For any $i,\, r,\, s=1,\,\ldots,\,n-1$ the relations

$$T^i_n(x)T^{-r}_{-s}(y)=T^{-r}_{-s}(y)T^i_n(x)-\delta^{i,r}h(y,x)\sum_{p=1}^{n-1}T^{-p}_{-s}(y)T^p_n(x)-\delta^{i,r}h(y,x)T^{-n}_{-s}(y)T^n_n(x).$$

hold. Since for every $w^{(-)}\in\tilde{\mathcal{W}}^{(-)}$

$$T^{-n}_{-s}(y)T^n_n(x)w^{(-)}=\lambda_n(x)T^{-n}_{-s}(y)w^{(-)}=0\,,$$

we see that for every $w^{(-)}\in\tilde{\mathcal{W}}^{(-)}$ and $i=1,\,\ldots,\,n-1$ we have $T^i_n(x)w^{(-)}=0$.
Since $\tilde{\mathcal{W}}=\mathcal{A}^{(+)}\tilde{\mathcal{W}}^{(-)}$, it is sufficient to show that if (4) holds for w, it also holds for $T^r_s(y)w$, where $r,s=1,\ldots,n-1$.
For $i=1,\,\ldots,\,n$ a $r,\,s=1,\,\ldots,\,n-1$ we have the commutation relation

$$T^i_n(x)T^r_s(y)=T^r_s(y)T^i_n(x)+g(y,x)T^i_s(y)T^r_n(x)-g(y,x)T^i_s(x)T^r_n(y)\,,$$

from which we can easily see that for any $w\in\tilde{\mathcal{W}}$

$$T^i_n(x)w=0\quad\text{for}\quad i=1,\,\ldots,\,n-1\,,\qquad T^n_n(x)w=\lambda_n(x)w$$

holds.
The relation $T^{-n}_{-n}(x)w=\lambda_{-n}(x)w$ results from the fact that for every $r,\,s=1,\,\ldots,\,n-1$ we have

$$T^{-n}_{-n}(x)T^r_s(y)=T^r_s(y)T^{-n}_{-n}(x)\,.$$

For $i,\,r,\,s=1,\,\ldots,\,n-1$ we use

$$T^{-n}_{-i}(x)T^r_s(y)=T^r_s(y)T^{-n}_{-i}(x)-\delta_{i,s}h(x,y)\sum_{p=1}^{n-1}T^r_p(y)T^{-n}_{-p}(x)-\delta_{i,s}h(x,y)T^r_n(y)T^{-n}_{-n}(x)\,,$$

which implies that $T^{-n}_{-i}(x)w=0$ for $i=1,\,\ldots,n-1$ and for any $w\in\tilde{\mathcal{W}}$. $\square$

Proposition 2. The space $\tilde{\mathcal{W}}$ is invariant with respect to $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$.
Proof: Obviously, the space $\tilde{\mathcal{W}}$ is invariant for the $\tilde{\mathcal{A}}^{(+)}$ action.
To show that the space $\tilde{\mathcal{W}}$ is invariant to the action of the algebra $\tilde{\mathcal{A}}^{(-)}$, we will use for $i,\,k,\,r,\,s=1,\,\ldots,\,n-1$ the commutation relations

$$\begin{aligned}T^{-i}_{-k}(x)T^r_s(y)-\delta_{k,s}k(y,x)\sum_{p=1}^{n-1}T^{-i}_{-p}(x)T^r_p(y)-\delta_{k,s}k(y,x)T^{-i}_{-n}(x)T^r_n(y)=\\=T^r_s(y)T^{-i}_{-k}(x)-\delta^{i,r}k(y,x)\sum_{p=1}^{n-1}T^p_s(y)T^{-p}_{-k}(x)-\delta^{i,r}k(y,x)T^n_s(y)T^{-n}_{-k}(x)\,.\end{aligned}$$

From Proposition 1 it follows that if we restrict these relations to subspace $\tilde{\mathcal{W}}$, we get

$$T^{-i}_{-k}(x)T^r_s(y)-\delta_{k,s}k(y,x)\sum_{p=1}^{n-1}T^{-i}_{-p}(x)T^r_p(y)=T^r_s(y)T^{-i}_{-k}(x)-\delta^{i,r}k(y,x)\sum_{p=1}^{n-1}T^p_s(y)T^{-p}_{-k}(x)\,.$$

If we multiply these equations by $\big(\delta^k_a\delta^s_b-\delta^{k,s}\delta_{a,b}\tilde{h}(x,y)\big)$, where

$$\tilde{h}(x,y)=\frac{1}{x-y+(n-1)-\eta}\,,$$

and sum them over k, s from 1 to $n-1$ and rename the indices, we find that the relations

$$T^{-i}_{-k}(x)T^r_s(y) = T^r_s(y)T^{-i}_{-k}(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n-1}T^p_s(y)T^{-p}_{-k}(x) - \\ -\delta_{k,s}\tilde{h}(x,y)\sum_{p=1}^{n-1}T^r_p(y)T^{-i}_{-p}(x) + \delta^{i,r}\delta_{k,s}\tilde{h}(x,y)k(y,x)\sum_{p,q=1}^{n-1}T^p_q(y)T^{-p}_{-q}(x)$$

are true on the space $\tilde{\mathcal{W}}$.
The invariance of the space $\tilde{\mathcal{W}}$ with respect to the action $\tilde{\mathcal{A}}^{(-)}$ can be proven by induction according to numbers of the factors $T^i_k(y)$ in the vectors $w \in \tilde{\mathcal{W}}$. □

Proposition 3. If we define

$$\tilde{\mathbf{T}}^{(+)}(x) = \sum_{i,k=1}^{n-1}\mathbf{E}^k_i \otimes T^i_k(x)\,, \qquad \tilde{\mathbf{T}}^{(-)}(x) = \sum_{i,k=1}^{n-1}\mathbf{E}^{-k}_{-i} \otimes T^{-i}_{-k}(x)$$

the commutation relations for $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where i, $k = 1, \ldots, n-1$, reduced to the space $\tilde{\mathcal{W}}$ can been written in the form of the RTT–equation

$$\tilde{\mathbf{R}}^{(\epsilon_1,\epsilon_2)}_{1,2}(x,y)\tilde{\mathbf{T}}^{(\epsilon_1)}_1(x)\tilde{\mathbf{T}}^{(\epsilon_2)}_2(y) = \tilde{\mathbf{T}}^{(\epsilon_2)}_2(y)\tilde{\mathbf{T}}^{(\epsilon_1)}_1(x)\tilde{\mathbf{R}}^{(\epsilon_1,\epsilon_2)}_{1,2}(x,y)$$

where ϵ_1, $\epsilon_2 = \pm$ and

$$\tilde{\mathbf{R}}^{(+,+)}_{1,2}(x,y) = \frac{1}{f(x,y)}\Big(\tilde{\mathbf{I}}_+ \otimes \tilde{\mathbf{I}}_+ + g(x,y)\sum_{i,k=1}^{n-1}\mathbf{E}^i_k \otimes \mathbf{E}^k_i\Big),$$
$$\tilde{\mathbf{R}}^{(-,-)}_{1,2}(x,y) = \frac{1}{f(x,y)}\Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_- + g(x,y)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i}\Big),$$
$$\tilde{\mathbf{R}}^{(+,-)}_{1,2}(x,y) = \tilde{\mathbf{I}}_+ \otimes \tilde{\mathbf{I}}_- - k(x,y)\sum_{i,k=1}^{n-1}\mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k},$$
$$\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y) = \tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - \tilde{h}(x,y)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k$$
$$\tilde{\mathbf{I}}_+ = \sum_{k=1}^{n-1}\mathbf{E}^k_k\,, \qquad \tilde{\mathbf{I}}_- = \sum_{k=1}^{n-1}\mathbf{E}^{-k}_{-k}\,, \qquad \tilde{h}(x,y) = \frac{1}{x-y+n-1-\eta}\,.$$

Proof: If we consider only the indices i, k, r, $s = 1, \ldots, n-1$ in the commutation relations

$$T^i_k(x)T^r_s(y) + g(x,y)T^r_k(x)T^i_s(y) = T^r_s(y)T^i_k(x) + g(x,y)T^r_k(y)T^i_s(x)$$
$$T^{-i}_{-k}(x)T^{-r}_{-s}(y) + g(x,y)T^{-r}_{-k}(x)T^{-i}_{-s}(y) = T^{-r}_{-s}(y)T^{-i}_{-k}(x) + g(x,y)T^{-r}_{-k}(y)T^{-i}_{-s}(x)$$

we can write them in the matrix form

$$\tilde{\mathbf{R}}^{(+,+)}_{1,2}(x,y)\tilde{\mathbf{T}}^{(+)}_1(x)\tilde{\mathbf{T}}^{(+)}_2(y) = \tilde{\mathbf{T}}^{(+)}_2(y)\tilde{\mathbf{T}}^{(+)}_1(x)\tilde{\mathbf{R}}^{(+,+)}_{1,2}(x,y)$$
$$\tilde{\mathbf{R}}^{(-,-)}_{1,2}(x,y)\tilde{\mathbf{T}}^{(-)}_1(x)\tilde{\mathbf{T}}^{(-)}_2(y) = \tilde{\mathbf{T}}^{(-)}_2(y)\tilde{\mathbf{T}}^{(-)}_1(x)\tilde{\mathbf{R}}^{(-,-)}_{1,2}(x,y)\,.$$

For i, k, r, $s = 1, \ldots, n-1$ we have in the RTT–algebra $\mathcal{A}_n$ the commutation relations

$$T^i_k(x)T^{-r}_{-s}(y) - \delta^{i,r}k(x,y)\sum_{p=1}^{n}T^p_k(x)T^{-p}_{-s}(y) = T^{-r}_{-s}(y)T^i_k(x) - \delta_{k,s}k(x,y)\sum_{p=1}^{n}T^{-r}_{-p}(y)T^i_p(x)$$
$$T^{-i}_{-k}(x)T^r_s(y) - \delta_{k,s}k(y,x)\sum_{p=1}^{n}T^{-i}_{-p}(x)T^r_p(y) = T^r_s(y)T^{-i}_{-k}(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n}T^p_s(y)T^{-p}_{-k}(x)\,.$$

If we restrict them on the space $\tilde{\mathcal{W}}$, we obtain according to Proposition 1

$$T^i_k(x)T^{-r}_{-s}(y) - \delta^{i,r}k(x,y)\sum_{p=1}^{n-1}T^p_k(x)T^{-p}_{-s}(y) = T^{-r}_{-s}(y)T^i_k(x) - \delta_{k,s}k(x,y)\sum_{p=1}^{n-1}T^{-r}_{-p}(y)T^i_p(x)$$
$$T^{-i}_{-k}(x)T^r_s(y) - \delta_{k,s}k(y,x)\sum_{p=1}^{n-1}T^{-i}_{-p}(x)T^r_p(y) = T^r_s(y)T^{-i}_{-k}(x) - \delta^{i,r}k(y,x)\sum_{p=1}^{n-1}T^p_s(y)T^{-p}_{-k}(x)\,.$$

The first of these commutation relations is

$$\tilde{\mathbf{R}}^{(+,-)}_{1,2}(x,y)\tilde{\mathbf{T}}^{(+)}_1(x)\tilde{\mathbf{T}}^{(-)}_2(y) = \tilde{\mathbf{T}}^{(-)}_2(y)\tilde{\mathbf{T}}^{(+)}_1(x)\tilde{\mathbf{R}}^{(+,-)}_{1,2}(x,y).$$

The second equality can be written using matrices in the form

$$\tilde{\mathbf{T}}^{(-)}_1(x)\tilde{\mathbf{T}}^{(+)}_2(y)\Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - k(y,x)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\Big) =$$
$$= \Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - k(y,x)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\Big)\tilde{\mathbf{T}}^{(+)}_2(y)\tilde{\mathbf{T}}^{(-)}_1(x).$$

And since

$$\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y)\Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - k(y,x)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\Big) =$$
$$= \Big(\tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+ - k(y,x)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\Big)\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y) = \tilde{\mathbf{I}}_- \otimes \tilde{\mathbf{I}}_+$$

this relation is equivalent to the RTT–equation

$$\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y)\tilde{\mathbf{T}}^{(-)}_1(x)\tilde{\mathbf{T}}^{(+)}_2(y) = \tilde{\mathbf{T}}^{(-)}_2(y)\tilde{\mathbf{T}}^{(+)}_1(x)\tilde{\mathbf{R}}^{(-,+)}_{1,2}(x,y)\,.$$

□

The following theorem immediately follows from Proposition 3.

Theorem 1. The action of the operators $T^i_k(x)$ and $T^{-i}_{-k}(x)$, where $i,\,k = 1,\,\ldots,\,n-1$, in the space $\tilde{\mathcal{W}}$ forms the RTT–algebra $\mathcal{A}_{n-1}$.

3 General form of common eigenvectors of $H^{(+)}(x)$ and $H^{(-)}(x)$

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Let $\vec{v} = (v_1, v_2, \ldots, v_P)$ and $\vec{w} = (w_1, w_2, \ldots, w_Q)$ be ordered sets of mutually different numbers. We will search for a general shape of the common eigenvectors $H^{(+)}(x)$ and $H^{(-)}(x)$ in the form

$$\mathfrak{B}(\vec{v},\vec{w}) = \sum_{k_1,\ldots,k_P=1}^{n-1}\ \sum_{r_1,\ldots,r_Q=1}^{n-1} T^n_{k_1}(v_1)\ldots T^n_{k_P}(v_P)T^{-r_1}_{-n}(w_1)\ldots T^{-r_Q}_{-n}(w_Q)\Phi^{k_1,\ldots,k_P}_{-r_1,\ldots,-r_Q}\,,$$

where $\Phi^{k_1,\ldots,k_P}_{-r_1,\ldots,-r_Q} \in \tilde{\mathcal{W}}$.

We will consider $(n-1)$–dimensional spaces $\mathcal{V}_+$ and $\mathcal{V}_-$ with the base $\mathbf{e}_k$ and $\mathbf{e}_{-r}$ and denote $\mathbf{f}^k$ and $\mathbf{f}^{-r}$ their dual base in dual spaces $\mathcal{V}_+^*$ and $\mathcal{V}_-^*$.
Let us define

$$\mathbf{b}^{(+)}(v) = \sum_{k=1}^{n-1} \mathbf{f}^k \otimes T_k^n(v) \in \mathcal{V}_+^* \otimes \mathcal{A}_n$$
$$\mathbf{b}^{(-)}(w) = \sum_{r=1}^{n-1} \mathbf{e}_{-r} \otimes T_{-n}^{-r}(w) \in \mathcal{V}_- \otimes \mathcal{A}_n$$

and denote

$$\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}) = \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{2^*}(v_2)\ldots\mathbf{b}^{(+)}_{P^*}(v_P) \in \mathcal{V}_1^* \otimes \mathcal{V}_2^* \otimes \ldots \otimes \mathcal{V}_P^* \otimes \mathcal{A}_n$$
$$\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}) = \mathbf{b}^{(-)}_1(w_1)\mathbf{b}^{(-)}_2(w_2)\ldots\mathbf{b}^{(-)}_Q(w_Q) \in \mathcal{V}_{-1} \otimes \mathcal{V}_{-2} \otimes \ldots \otimes \mathcal{V}_{-Q} \otimes \mathcal{A}_n\,.$$

Explicitly, we have

$$\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}) = \sum_{k_1,\ldots,k_P=1}^{n-1} \mathbf{f}^{k_1} \otimes \mathbf{f}^{k_2} \otimes \ldots \otimes \mathbf{f}^{k_P} \otimes T^n_{k_1}(v_1)T^n_{k_2}(v_2)\ldots T^n_{k_P}(v_P)$$
$$\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}) = \sum_{r_1,\ldots,r_Q=1}^{n-1} \mathbf{e}_{-r_1} \otimes \mathbf{e}_{-r_2} \otimes \ldots \otimes \mathbf{e}_{-r_Q} \otimes T^{-r_1}_{-n}(w_1)T^{-r_2}_{-n}(w_2)\ldots T^{-r_Q}_{-n}(w_Q)\,.$$

If we introduce $\mathbf{\Phi} \in \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_P \otimes \mathcal{V}_{-1}^* \otimes \cdots \otimes \mathcal{V}_{-Q}^* \otimes \tilde{W}$

$$\mathbf{\Phi} = \sum_{k_1,\ldots,k_P=1}^{n-1} \sum_{r_1,\ldots,r_Q=1}^{n-1} \mathbf{e}_{k_1} \otimes \ldots \otimes \mathbf{e}_{k_P} \otimes \mathbf{f}^{-r_1} \otimes \ldots \otimes \mathbf{f}^{-r_Q} \otimes \Phi^{k_1,k_2,\ldots,k_P}_{-r_1,-r_2,\ldots,-r_Q} =$$
$$= \sum_{\vec{k},\vec{r}} \mathbf{e}_{\vec{k}} \otimes \mathbf{f}^{-\vec{r}} \otimes \Phi^{\vec{k}}_{-\vec{r}}\,,$$

where

$$\Phi^{k_1,k_2,\ldots,k_P}_{-r_1,-r_2,\ldots,-r_Q} = \Phi^{\vec{k}}_{-\vec{r}} \in \tilde{\mathcal{W}}\,,$$
$$\mathbf{e}_{\vec{k}} = \mathbf{e}_{k_1} \otimes \mathbf{e}_{k_2} \otimes \ldots \otimes \mathbf{e}_{k_P} \in \big(\mathcal{V}_+\big)^{\otimes P},$$
$$\mathbf{f}^{-\vec{r}} = \mathbf{f}^{-r_1} \otimes \mathbf{f}^{-r_2} \otimes \ldots \otimes \mathbf{f}^{-r_Q} \in \big(\mathcal{V}_-^*\big)^{\otimes Q},$$

the assumed shape of the eigenvectors can be written as

$$\mathfrak{B}(\vec{v},\vec{w}) = \Big\langle \mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}), \mathbf{\Phi} \Big\rangle.$$

4 Bethe vectors and Bethe condition

Our goal is to write the action of the operators $T_n^n(x)$, $T_{-n}^{-n}(x)$, $\tilde{\mathbf{T}}^{(+)}$ and $\tilde{\mathbf{T}}^{(-)}$ on the assumed form of the Bethe vectors using the operators that act only on $\mathbf{\Phi}$. These actions are explicitly given in Lemma 5 of Appendix. Here we list only their consequences.
For $\vec{v} = (v_1, v_2, \ldots, v_P)$ we introduce a set $\overline{v} = \{v_1, v_2, \ldots, v_P\}$, denote

$$\vec{v}_k = (v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_P)\,, \qquad \overline{v}_k = \overline{v} \setminus \{v_k\}\,,$$
$$F(x;\overline{v}) = \prod_{v_k \in \overline{v}} f(x, v_k)\,, \qquad F(\overline{v};x) = \prod_{v_k \in \overline{v}} f(v_k, x)\,.$$

and define

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w}) = \widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\tilde{\mathbf{T}}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(x;\vec{v}) = \sum_{i,k=1}^{n-1}\mathbf{E}^k_i\otimes\widehat{T}^i_k(x;\vec{v};\vec{w})$$
$$\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\tilde{\mathbf{T}}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v}) = \sum_{i,k=1}^{n-1}\mathbf{E}^{-k}_{-i}\otimes\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\,,$$

where

$$\begin{aligned}
&\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(x;\vec{v}) = \widehat{\mathbf{R}}^{(+,+)}_{0_+,P_+}(x,v_P)\dots\widehat{\mathbf{R}}^{(+,+)}_{0_+,2_+}(x,v_2)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\\
&\widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w}) = \widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(+,-)}_{0_+,2^*_-}(x,w_2)\dots\widehat{\mathbf{R}}^{(+,-)}_{0_+,Q^*_-}(x,w_Q)\\
&\widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(x;\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(x,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{0_-,Q^*_-}(x,w_Q)\\
&\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v}) = \widehat{\mathbf{R}}^{(-,+)}_{0_-,P_+}(x,v_P)\dots\widehat{\mathbf{R}}^{(-,+)}_{0_-,2_+}(x,v_2)\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v_1)\\
&\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v) = \frac{1}{f(x,v)}\Big(\tilde{\mathbf{I}}_+\otimes\tilde{\mathbf{I}}_+ + g(x,v)\sum_{i,k=1}^{n-1}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\Big)\\
&\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w) = \tilde{\mathbf{I}}_+\otimes\tilde{\mathbf{I}}^*_- - \tilde{h}(w,x)\sum_{r,s=1}^{n-1}\mathbf{E}^r_s\otimes\mathbf{F}^{-r}_{-s}\\
&\tilde{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v) = \tilde{\mathbf{I}}_-\otimes\tilde{\mathbf{I}}_+ - \tilde{h}(x,v)\sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k\\
&\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w) = \frac{1}{f(w,x)}\Big(\tilde{\mathbf{I}}_{0_-}\otimes\tilde{\mathbf{I}}^*_{1_-} + g(w,x)\sum_{r,s=1}^{n-1}\mathbf{E}^{-r}_{-s}\otimes\mathbf{F}^{-s}_{-r}\Big).
\end{aligned}$$

The main results of this paper are the following three Theorems.

Theorem 2. Let $\mathbf{\Phi} = \sum_{\vec{k},\vec{r}}\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\otimes\Phi^{\vec{k}}_{-\vec{r}}$, where $\Phi^{\vec{k}}_{-\vec{r}}\in\tilde{\mathcal{W}}$ is an common eigenvector of the operators

$$\widehat{H}^{(+)}(x;\vec{v};\vec{w}) = \mathrm{Tr}_0\big(\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\big) = \sum_{i=1}^{n-1}\widehat{T}^i_i(x;\vec{v};\vec{w})$$
$$\widehat{H}^{(-)}(x;\vec{v};\vec{w}) = \mathrm{Tr}_0\big(\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\big) = \sum_{i=1}^{n-1}\widehat{T}^{-i}_{-i}(x;\vec{v};\vec{w})$$

with eigenvalues $\mu^{(+)}(x;\vec{v};\vec{w})$ and $\mu^{(-)}(x;\vec{v};\vec{w})$. If the Bethe conditions

$$\begin{aligned}
&\lambda_n(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta) = \mu^{(+)}(v_\ell;\vec{v};\vec{w})F(v_\ell;\overline{v}_\ell)\\
&\lambda_{-n}(w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v}) = \mu^{(-)}(w_s;\vec{v};\vec{w})F(\overline{w}_s;w_s)
\end{aligned}\tag{5}$$

are fulfilled for any $v_\ell\in\overline{v}$ and $w_s\in\overline{w}$, the vector

$$\mathfrak{B}(\vec{v};\vec{w}) = \Big\langle\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{\Phi}\Big\rangle$$

is a common eigenvector of the operators $H^{(+)}(x)$ and $H^{(-)}(x)$ with eigenvalues

$$\begin{aligned}
&E^{(+)}(x;\vec{v};\vec{w}) = \lambda_n(x)F(\overline{v};x)F(\overline{w};x-n+1+\eta) + \mu^{(+)}(x;\vec{v},\vec{w})F(x;\overline{v})\\
&E^{(-)}(x;\vec{v};\vec{w}) = \lambda_{-n}(x)F(x;\overline{w})F(x+n-1-\eta;\overline{v}) + \mu^{(-)}(x;\vec{v};\vec{w})F(\overline{w};x)\,.
\end{aligned}$$

Proof: According to Proposition 1, we have $T^n_n(x)\mathbf{\Phi} = \lambda_n(x)\mathbf{\Phi}$ and $T^{-n}_{-n}(x)\mathbf{\Phi} = \lambda_{-n}(x)\mathbf{\Phi}$.

Using Lemma 5 and the relations $\mathrm{Tr}_0\, \widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-} = \widehat{\mathbf{I}}_{s^*_-}$ and $\mathrm{Tr}_0\, \widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+} = \widehat{\mathbf{I}}_{\ell_-}$ we obtain

$$
\begin{aligned}
&H^{(+)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \mathbf{\Phi}\Big\rangle = \\
&\quad = \Big(\lambda_n(x)F(\overline{v};x)F(\overline{w};x-n+1+\eta) + \mu^{(+)}(x;\vec{v},\vec{w})F(x;\overline{v})\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \mathbf{\Phi}\Big\rangle - \\
&\quad - \sum_{v_\ell\in\overline{v}} g(v_\ell,x)\Big(\lambda_n(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta) - \mu^{(+)}(v_\ell;\vec{v};\vec{w})F(v_\ell;\overline{v}_\ell)\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{\Phi}\Big\rangle - \\
&\quad - \sum_{w_s\in\overline{w}} \tilde{h}(w_s,x)\Big(\lambda_{-n}(w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v}) - \mu^{(-)}(w_s;\vec{v};\vec{w})F(\overline{w}_s;w_s)\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec{w}_s), \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{\Phi}\Big\rangle \\
&H^{(-)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \mathbf{\Phi}\Big\rangle = \\
&\quad = \Big(\lambda_{-n}(x)F(x;\overline{w})F(x+n-1-\eta;\overline{v}) + \mu^{(-)}(x;\vec{v};\vec{w})F(\overline{w};x)\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \mathbf{\Phi}\Big\rangle - \\
&\quad - \sum_{w_s\in\overline{w}} g(x,w_s)\Big(\lambda_{-n}(w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v}) - \mu^{(-)}(w_s;\vec{v},\vec{w})F(\overline{w}_s;w_s)\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s), \widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{\Phi}\Big\rangle - \\
&\quad - \sum_{v_\ell\in\overline{v}} \tilde{h}(x,v_\ell)\Big(\lambda_n(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta) - \mu^{(+)}(v_\ell;\vec{v};\vec{w})F(v_\ell;\overline{v}_\ell)\Big) \\
&\qquad\qquad\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{\Phi}\Big\rangle.
\end{aligned}
$$

This immediately proves the statement of Theorem 2. □

Theorem 3. The operators $\widehat{T}^{i}_{k}(x;\vec{v};\vec{w})$ and $\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})$ are for any $\vec{v}$ and $\vec{w}$ generators of the RTT–algebra of $\mathcal{A}_{n-1}$ type.

PROOF: We have to show that for any $\vec{v}$, $\vec{w}$ and ϵ, $\epsilon' = \pm$ the relation

$$
\begin{aligned}
&\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,P;1^*,\dots,Q^*}(y;\vec{v};\vec{w}) = \\
&\qquad\qquad = \widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,P;1^*,\dots,Q^*}(y;\vec{v};\vec{w})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\,.
\end{aligned}
$$

is valid. Since

$$
\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w}) = \widehat{\mathbf{R}}^{(\epsilon,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\tilde{\mathbf{T}}^{(\epsilon)}_{0}(x)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1,\dots,P}(x;\vec{v})
$$

and $\tilde{\mathbf{T}}^{(\epsilon)}_0(x)$ satisfies the RTT–equation, it is enough to show that

$$
\begin{aligned}
&\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\widehat{\mathbf{R}}^{(\epsilon,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\widehat{\mathbf{R}}^{(\epsilon',-)}_{0';1^*,\dots,Q^*}(y;\vec{w}) = \widehat{\mathbf{R}}^{(\epsilon',-)}_{0';1^*,\dots,Q^*}(y;\vec{w})\widehat{\mathbf{R}}^{(\epsilon,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\tilde{\mathbf{R}}^{(\epsilon,\epsilon;)}_{0,0'}(x,y) \\
&\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1,\dots,P}(x;\vec{v})\widehat{\mathbf{R}}^{(\epsilon',+)}_{0';1,\dots,P}(y;\vec{v}) = \widehat{\mathbf{R}}^{(\epsilon',+)}_{0';1,\dots,P}(y;\vec{v})\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1,\dots,P}(x;\vec{v})\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\,.
\end{aligned}
$$

hold. According to the definitions, we have

$$
\begin{aligned}
&\widehat{\mathbf{R}}^{(\epsilon,-)}_{0;1^*,\dots,Q^*}(x;\vec{w}) = \widehat{\mathbf{R}}^{(\epsilon,-)}_{0_\epsilon,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(\epsilon,-)}_{0_\epsilon,2^*_-}(x,w_2)\dots\widehat{\mathbf{R}}^{(\epsilon,-)}_{0_\epsilon,Q^*_-}(x,w_Q) \\
&\widehat{\mathbf{R}}^{(\epsilon,+)}_{0;1,\dots,P}(x;\vec{v}) = \widehat{\mathbf{R}}^{(\epsilon,+)}_{0_\epsilon,P_+}(x,v_P)\dots\widehat{\mathbf{R}}^{(\epsilon,+)}_{0_\epsilon,2_+}(x,v_2)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0_\epsilon,1_+}(x,v_1)\,.
\end{aligned}
$$

and a theorem then follows from the Yang–Baxter equations

$$\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\widehat{\mathbf{R}}^{(\epsilon,-)}_{0_\epsilon,1^*_-}(x,w)\widehat{\mathbf{R}}^{(\epsilon',-)}_{0'_{\epsilon'},1^*_-}(y,w) = \widehat{\mathbf{R}}^{(\epsilon',-)}_{0'_{\epsilon'},1^*_-}(y,w)\widehat{\mathbf{R}}^{(\epsilon,-)}_{0_\epsilon,1^*_-}(x,w)\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\,,$$
$$\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0_\epsilon,1_+}(x,v)\widehat{\mathbf{R}}^{(\epsilon',+)}_{0'_{\epsilon'},1_+}(y,v) = \widehat{\mathbf{R}}^{(\epsilon',+)}_{0'_{\epsilon'},1_+}(y,v)\widehat{\mathbf{R}}^{(\epsilon,+)}_{0_\epsilon,1_+}(x,v)\tilde{\mathbf{R}}^{(\epsilon,\epsilon')}_{0,0'}(x,y)\,.$$

□

The following Theorem shows that

$$\widehat{\mathbf{\Omega}} = \underbrace{\mathbf{e}_{n-1}\otimes\ldots\otimes\mathbf{e}_{n-1}}_{P\times}\otimes\underbrace{\mathbf{f}^{-n+1}\otimes\ldots\otimes\mathbf{f}^{-n+1}}_{Q\times}\otimes\omega$$

is a vacuum vector for the representation of the RTT–algebra $\mathcal{A}_{n-1}$, which is generated by $\widehat{T}^i_k(x;\vec{v};\vec{w})$ and $\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})$.

Theorem 4. For the vector $\widehat{\mathbf{\Omega}}$ and $i,\,k=1,\,\ldots,\,n-1$

$$\widehat{T}^i_k(x;\vec{v},\vec{w})\widehat{\mathbf{\Omega}}=0 \quad \text{for} \quad i<k\,, \qquad \widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=0 \quad \text{for} \quad k<i$$
$$\widehat{T}^i_i(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=\nu_i(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} \qquad \widehat{T}^{-i}_{-i}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=\nu_{-i}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}$$

where

$$\nu_i(x;\vec{v};\vec{w}) = \lambda_i(x)F(\overline{v};x+1) \qquad \text{for} \quad 1\le i<n-1$$
$$\nu_{n-1}(x;\vec{v};\vec{w}) = \lambda_{n-1}(x)F(x-n+1+\eta;\overline{w})$$
$$\nu_{-i}(x;\vec{v};\vec{w}) = \lambda_{-i}(x)F(x-1;\overline{w}) \qquad \text{for} \quad 1\le i<n-1$$
$$\nu_{-n+1}(x;\vec{v};\vec{w}) = \lambda_{-n+1}(x)F(\overline{v};x+n-1-\eta)$$

are valid.
Proof: If we write

$$\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v) = \sum_{a,b,p,q=1}^{n-1} R^{a,p}_{b,q}(x,v)\mathbf{E}^b_a\otimes\mathbf{E}^q_p \qquad R^{a,p}_{b,q}(x,v) = \frac{\delta^a_b\delta^p_q + g(x,v)\delta^a_q\delta^p_b}{f(x,v)}$$
$$\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w) = \sum_{c,d,r,s=1}^{n-1} R^{c,-r}_{d,-s}(x,w)\mathbf{E}^d_c\otimes\mathbf{F}^{-s}_{-r} \qquad R^{c,-r}_{d,-s}(x,w) = \delta^c_d\delta^r_s - \tilde{h}(w,x)\delta^{c,r}\delta_{d,s}\,,$$

$$\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v) = \sum_{a,b,p,q=1}^{n-1} R^{-a,p}_{-b,q}(x,v)\mathbf{E}^{-b}_{-a}\otimes\mathbf{E}^q_p \qquad R^{-a,p}_{-b,q}(x,v) = \delta^a_b\delta^p_q - \tilde{h}(x,v)\delta^{a,p}\delta_{b,q}$$
$$\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w) = \sum_{c,d,r,s=1}^{n-1} R^{-c,-r}_{-d,-s}(x,w)\mathbf{E}^{-d}_{-c}\otimes\mathbf{F}^{-s}_{-r} \qquad R^{-c,-r}_{-d,-s}(x,w) = \frac{\delta^c_d\delta^r_s + g(w,x)\delta^c_s\delta^r_d}{f(w,x)}\,,$$

we obtain

$$\widehat{T}^i_k(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} =$$
$$= R^{i,-n+1}_{d_1,-s_1}(x,w_1)R^{d_1,-n+1}_{d_2,-s_2}(x,w_2)\dots R^{d_{Q-2},-n+1}_{d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{d_{Q-1},-n+1}_{d_Q,-s_Q}(x,w_Q)$$
$$R^{a_P,p_P}_{a_{P-1},n-1}(x,v_P)R^{a_{P-1},p_{P-1}}_{a_{P-2},n-1}(x,v_{P-1})\dots R^{a_3,p_2}_{a_1,n-1}(x,v_2)R^{a_1,p_1}_{k,n-1}(x,v_1)$$
$$\mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\dots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes$$
$$\otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{d_Q}_{a_P}(x)\omega$$

$$\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} =$$
$$= R^{-i,-n+1}_{-d_1,-s_1}(x,w_1)R^{-d_1,-n+1}_{-d_2,-s_2}(x,w_2)\dots R^{-d_{Q-2},-n+1}_{-d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{-d_{Q-1},-n+1}_{-d_Q,-s_Q}(x,w_Q)$$
$$R^{-a_P,p_P}_{-a_{P-1},n-1}(x,v_P)R^{-a_{P-1},p_{P-1}}_{-a_{P-2},n-1}(x,v_{P-1})\dots R^{-a_2,p_2}_{-a_1,n-1}(x,v_2)R^{-a_1,p_1}_{-k,n-1}(x,v_1)$$
$$\mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\dots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes$$
$$\otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{-d_Q}_{-a_P}(x)\omega$$

Since $R^{i,-n+1}_{d,-s}(x,w)=\delta^i_d\delta^{n-1}_s$ for $1\le i<n-1$, we have

$$\widehat{T}^i_k(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = R^{a_P,p_P}_{a_{P-1},n-1}(x,v_P)R^{a_{P-1},p_{P-1}}_{a_{P-2},n-1}(x,v_{P-1})\dots R^{a_3,p_2}_{a_1,n-1}(x,v_2)R^{a_1,p_1}_{k,n-1}(x,v_1)$$
$$\mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\dots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes$$
$$\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\dots\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes T^i_{a_P}(x)\omega$$

As $T^i_{a_P}(x)\omega=0$ for $a_P>i$, this expression is nonzero only for $a_P\le i<n-1$. In this case $R^{a_P,p_P}_{a_{P-1},n-1}(x,v_P)=\dfrac{1}{f(x,v_P)}\,\delta^{a_P}_{a_{P-1}}\delta^{p_P}_{n-1}$. Therefore, we have

$$\widehat{T}^i_k(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \frac{1}{F(x;\overline{v})}\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes$$
$$\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\dots\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes T^i_k(x)\omega$$

and so

$$\widehat{T}^i_k(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=0 \quad\text{for}\quad k>i\,,\qquad \widehat{T}^i_i(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=\frac{\lambda_i(x)}{F(x;\overline{v})}\,\widehat{\mathbf{\Omega}}=\lambda_i(x)F(\overline{v};x+1)\widehat{\mathbf{\Omega}}\,.$$

If $i=k=n-1$ we have

$$\widehat{T}^{n-1}_{n-1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} =$$
$$= R^{n-1,-n+1}_{d_1,-s_1}(x,w_1)R^{d_1,-n+1}_{d_2,-s_2}(x,w_2)\dots R^{d_{Q-2},-n+1}_{d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{d_{Q-1},-n+1}_{d_Q,-s_Q}(x,w_Q)$$
$$R^{a_P,p_P}_{a_{P-1},n-1}(x,v_P)R^{a_{P-1},p_{P-1}}_{a_{P-2},n-1}(x,v_{P-1})\dots R^{a_3,p_2}_{a_1,n-1}(x,v_2)R^{a_1,p_1}_{n-1,n-1}(x,v_1)$$
$$\mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\dots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes$$
$$\otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{d_Q}_{a_P}(x)\omega$$

Since $R^{a,p}_{n-1,n-1}(x,v)=\delta^a_{n-1}\delta^p_{n-1}$ we obtain

$$\widehat{T}^{n-1}_{n-1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} =$$
$$= R^{n-1,-n+1}_{d_1,-s_1}(x,w_1)R^{d_1,-n+1}_{d_2,-s_2}(x,w_2)\dots R^{d_{Q-2},-n+1}_{d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{d_{Q-1},-n+1}_{d_Q,-s_Q}(x,w_Q)$$
$$\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes$$
$$\otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{d_Q}_{n-1}(x)\omega\,.$$

The conditions $T^{d_Q}_{n-1}(x)\omega = 0$ for $d_Q < n-1$ and $T^{n-1}_{n-1}(x)\omega = \lambda_{n-1}(x)\omega$ lead to the equations

$$\begin{aligned}
&\widehat{T}^{n-1}_{n-1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \\
&\quad = \lambda_{n-1}(x)R^{n-1,-n+1}_{d_1,-s_1}(x,w_1)R^{d_1,-n+1}_{d_2,-s_2}(x,w_2)\dots R^{d_{Q-2},-n+1}_{d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{d_{Q-1},-n+1}_{n-1,-s_Q}(x,w_Q) \\
&\qquad \mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes \\
&\qquad\quad \otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes\omega
\end{aligned}$$

However, $R^{d,-n+1}_{n-1,-s}(x,w) = \big(1-\tilde{h}(w,x)\big)\delta^d_{n-1}\delta^{n-1}_s = f(x-n+1+\eta,w)\delta^d_{n-1}\delta^{n-1}_s$, and so

$$\widehat{T}^{n-1}_{n-1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \lambda_{n-1}(x)F(x-n+1+\eta;\overline{w})\widehat{\mathbf{\Omega}}.$$

Since $R^{-a,p}_{-k,n-1}(x,v) = \delta^a_k\delta^p_{n-1}$ for $1\le k<n-1$ we have

$$\begin{aligned}
&\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \\
&\quad = R^{-i,-n+1}_{-d_1,-s_1}(x,w_1)R^{-d_1,-n+1}_{-d_2,-s_2}(x,w_2)\dots R^{-d_{Q-2},-n+1}_{-d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{-d_{Q-1},-n+1}_{-d_Q,-s_Q}(x,w_Q) \\
&\qquad \mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes \\
&\qquad\quad \otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{-d_Q}_{-k}(x)\omega
\end{aligned}.$$

For $k<d_Q$ the relation $T^{-d_Q}_{-k}(x)\omega = 0$ holds. Therefore, this expression is nonzero for $1\le d_Q\le k<n-1$ only. But in this case $R^{-d_{Q-1},-n+1}_{-d_Q,-s_Q}(x,w_Q) = \dfrac{1}{f(w_Q,x)}\,\delta^{d_{Q-1}}_{\delta_Q}\delta^{n-1}_{s_Q}$. By repeatedly using this relationship, we get

$$\begin{aligned}
\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \frac{1}{F(\overline{w};x)}\,&\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes\dots\otimes\mathbf{e}_{n-1}\otimes\mathbf{e}_{n-1}\otimes \\
&\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\dots\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes T^{-i}_{-k}(x)\omega
\end{aligned}$$

The relations $T^{-i}_{-k}(x)\omega = 0$ for $k<i$ and $T^{-i}_{-i}(x)\omega = \lambda_{-i}(x)\omega$ lead to the equations $\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = 0$ for $k<i$ and

$$\widehat{T}^{-i}_{-i}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \frac{\lambda_{-i}(x)}{F(\overline{w};x)}\,\widehat{\mathbf{\Omega}} = \lambda_{-i}(x)F(x-1;\overline{w})\widehat{\mathbf{\Omega}}\,.$$

For $i=k=n-1$ we have

$$\begin{aligned}
&\widehat{T}^{-n+1}_{-n+1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}} = \\
&\quad = R^{-n+1,-n+1}_{-d_1,-s_1}(x,w_1)R^{-d_1,-n+1}_{-d_2,-s_2}(x,w_2)\dots R^{-d_{Q-2},-n+1}_{-d_{Q-1},-s_{Q-1}}(x,w_{Q-1})R^{-d_{Q-1},-n+1}_{-d_Q,-s_Q}(x,w_Q) \\
&\qquad R^{-a_P,p_P}_{-a_{P-1},n-1}(x,v_P)R^{-a_{P-1},p_{P-1}}_{-a_{P-2},n-1}(x,v_{P-1})\dots R^{-a_2,p_2}_{-a_1,n-1}(x,v_2)R^{-a_1,p_1}_{-n+1,n-1}(x,v_1) \\
&\qquad\quad \mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\dots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes \\
&\qquad\qquad \otimes\mathbf{f}^{-s_1}\otimes\mathbf{f}^{-s_2}\otimes\dots\otimes\mathbf{f}^{-s_{Q-1}}\otimes\mathbf{f}^{-s_Q}\otimes T^{-d_Q}_{-a_P}(x)\omega
\end{aligned}$$

Since $R^{-n+1,-n+1}_{-d,-s}(x,w)=\delta^{n-1}_d\delta^{n-1}_s$, we obtain

$$\widehat{T}^{-n+1}_{-n+1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}= \\ =R^{-a_P,p_P}_{-a_{P-1},n-1}(x,v_P)R^{-a_{P-1},p_{P-1}}_{-a_{P-2},n-1}(x,v_{P-1})\ldots R^{-a_2,p_2}_{-a_1,n-1}(x,v_2)R^{-a_1,p_1}_{-n+1,n-1}(x,v_1) \\ \mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\ldots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes \\ \otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\ldots\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes T^{-n+1}_{-a_P}(x)\omega$$

The conditions $T^{-n+1}_{-a_P}(x)\omega=0$ for $a_P<n-1$ and $T^{-n+1}_{-n+1}(x)\omega=\lambda_{-n+1}(x)\omega$ lead to

$$\widehat{T}^{-n+1}_{-n+1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}= \\ =\lambda_{-n+1}(x)R^{-n+1,p_P}_{-a_{P-1},n-1}(x,v_P)R^{-a_{P-1},p_{P-1}}_{-a_{P-2},n-1}(x,v_{P-1})\ldots R^{-a_2,p_2}_{-a_1,n-1}(x,v_2)R^{-a_1,p_1}_{-n+1,n-1}(x,v_1) \\ \mathbf{e}_{p_1}\otimes\mathbf{e}_{p_2}\otimes\ldots\otimes\mathbf{e}_{p_{P-1}}\otimes\mathbf{e}_{p_P}\otimes \\ \otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\ldots\otimes\mathbf{f}^{-n+1}\otimes\mathbf{f}^{-n+1}\otimes\omega$$

However, $R^{-n+1,p}_{-a,n-1}(x,v)=\big(1-\tilde{h}(x,v)\big)\delta^{n-1}_a\delta^p_{n-1}=f(v,x+n-1-\eta)\delta^{n-1}_a\delta^p_{n-1}$, and so

$$\widehat{T}^{-n+1}_{-n+1}(x;\vec{v};\vec{w})\widehat{\mathbf{\Omega}}=\lambda_{-n+1}(x)F(\overline{v};x+n-1-\eta)\widehat{\mathbf{\Omega}}$$

□

These three theorems show that to find the Bethe vectors $\mathfrak{B}(\vec{v};\vec{w})$ for the RTT–algebra $\mathcal{A}_n$, it is sufficient to find the Bethe vectors for the RTT–algebra $\mathcal{A}_{n-1}$ that is generated by the operators $\widehat{T}^i_k(x;\vec{v};\vec{w})$, $\widehat{T}^{-i}_{-k}(x;\vec{v};\vec{w})$, where $i,k=1,\ldots,n-1$, and that has a vacuum vector $\widehat{\mathbf{\Omega}}$.

5 Conclusion

The paper describes the construction of eigenvectors for the representations of the RTT–algebra $\mathcal{A}_n$ by using the highest weight vectors for the representation of the RTT–algebra $\mathcal{A}_{n-1}$. We meet these RTT–algebras [1, 2] while studying the algebraic Bethe ansatz for the RTT–algebras of sp($2n$) and o($2n$) types.

In the special cases, when $\overline{v}$ or $\overline{w}$ is an empty set, our construction is known as the algebraic nested Bethe ansatz, which was formulated in [3]. So our construction of the Bethe vectors is a generalization of the algebraic nested Bethe ansatz to the RTT–algebra of $\mathcal{A}_n$ type.

For the RTT–algebra of $\mathcal{A}_2$ type we get from theorems 2, 3 and 4 the Bethe vectors

$$\mathfrak{B}_2(\vec{v};\vec{w})=T^2_1(\vec{v})T^{-1}_{-2}(\vec{w})\omega$$

and the Bethe conditions

$$\lambda_2(v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-1+\eta)=\lambda_1(v_\ell)F(v_\ell-1+\eta;\overline{w})F(v_\ell;\overline{v}_\ell) \\ \lambda_{-2}(w_s)F(w_s;\overline{w}_s)F(w_s+1-\eta;\overline{v})=\lambda_{-1}(w_s)F(\overline{v};w_s+1-\eta)\,F(\overline{w}_s;w_s),$$

which we found for this algebra and $\nu=-1$ in [4].

For higher n it is possible by means of Theorems 2, 3 and 4 step-by-step to decrease value n and thus obtain an explicit form of the Bethe vectors. For the RTT–algebra of gl(n) type this procedure leads to trace-formula [5]. We intend to publish a similar explicit form of the Bethe vectors for the RTT–algebras $\mathcal{A}_n$, of sp$(2n)$ and o$(2n)$ types in the near future.

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Appendix

A1 Commutation relations in the RTT–algebra $\mathcal{A}_n$

The RTT–equation for the RTT–algebra $\mathcal{A}_n$ leads to the commutation relations

$$
\begin{aligned}
&T^i_k(x)T^r_s(y)+g(x,y)T^r_k(x)T^i_s(y)=T^r_s(y)T^i_k(x)+g(x,y)T^r_k(y)T^i_s(x)\\
&T^{-i}_{-k}(x)T^{-r}_{-s}(y)+g(x,y)T^{-r}_{-k}(x)T^{-i}_{-s}(y)=T^{-r}_{-s}(y)T^{-i}_{-k}(x)+g(x,y)T^{-r}_{-k}(y)T^{-i}_{-s}(x)\\
&T^i_k(x)T^{-r}_{-s}(y)-\delta^{i,r}k(x,y)\sum_{p=1}^{n}T^p_k(x)T^{-p}_{-s}(y)=T^{-r}_{-s}(y)T^i_k(x)-\delta_{k,s}k(x,y)\sum_{p=1}^{n}T^{-r}_{-p}(y)T^i_p(x)\\
&T^{-i}_{-k}(x)T^r_s(y)-\delta^{i,r}h(x,y)\sum_{p=1}^{n}T^{-p}_{-k}(x)T^p_s(y)=T^r_s(y)T^{-i}_{-k}(x)-\delta_{k,s}h(x,y)\sum_{p=1}^{n}T^r_p(y)T^{-i}_{-p}(x)\\
&T^i_k(x)T^r_s(y)+g(y,x)T^i_s(x)T^r_k(y)=T^r_s(y)T^i_k(x)+g(y,x)T^i_s(y)T^r_k(x)\\
&T^{-i}_{-k}(x)T^{-r}_{-s}(y)+g(y,x)T^{-i}_{-s}(x)T^{-r}_{-k}(y)=T^{-r}_{-s}(y)T^{-i}_{-k}(x)+g(y,x)T^{-i}_{-s}(y)T^{-r}_{-k}(x)\\
&T^i_k(x)T^{-r}_{-s}(y)-\delta_{k,s}h(y,x)\sum_{p=1}^{n}T^i_p(x)T^{-r}_{-p}(y)=T^{-r}_{-s}(y)T^i_k(x)-\delta^{i,r}h(y,x)\sum_{p=1}^{n}T^{-p}_{-s}(y)T^p_k(x)\\
&T^{-i}_{-k}(x)T^r_s(y)-\delta_{k,s}k(y,x)\sum_{p=1}^{n}T^{-i}_{-p}(x)T^r_p(y)=T^r_s(y)T^{-i}_{-k}(x)-\delta^{i,r}k(y,x)\sum_{p=1}^{n}T^p_s(y)T^{-p}_{-k}(x)
\end{aligned}
$$

A2 Action of the operators $T^{\pm n}_{\pm n}(x)$ and $\tilde{\mathbf{T}}^{(\pm)}(x)$ on the Bethe vectors

First, we will rewrite the commutation relations using the operators action for $P=Q=1$.

Lemma 1. In the RTT–algebra $\mathcal{A}_n$ the following relations are true:

$$
\begin{aligned}
&T^n_n(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle = f(v,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle T^n_n(x) - g(v,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\mathbf{e}_k\Big\rangle T^n_n(v)\\
&T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle = f(x,w)\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle T^{-n}_{-n}(x) - g(x,w)\Big\langle \mathbf{b}^{(-)}_{1}(x),\mathbf{f}^{-r}\Big\rangle T^{-n}_{-n}(w)\\
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle = f(x,v)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\tilde{\mathbf{T}}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v)\big(\mathbf{I}_{0_+}\otimes\mathbf{e}_k\big)\Big\rangle-\\
&\qquad\qquad -g(x,v)\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\tilde{\mathbf{T}}^{(+)}_0(v)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\big(\mathbf{I}_{0_+}\otimes\mathbf{e}_k\big)\Big\rangle\\
&\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle = f(w,x)\Big\langle \mathbf{b}^{(-)}_{1}(w),\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w)\tilde{\mathbf{T}}^{(-)}_0(x)\big(\tilde{\mathbf{I}}_-\otimes\mathbf{f}^{-r}\big)\Big\rangle-\\
&\qquad\qquad -g(w,x)\Big\langle \mathbf{b}^{(-)}_{1}(x),\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\tilde{\mathbf{T}}^{(-)}_0(w)\big(\tilde{\mathbf{I}}_-\otimes\mathbf{f}^{-r}\big)\Big\rangle\\
&T^n_n(x)\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle = \frac{\tilde{h}(w,x)}{h(w,x)}\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle T^n_n(x)+\\
&\qquad\qquad +\tilde{h}(w,x)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\tilde{\mathbf{T}}^{(-)}_0(w)\big(\tilde{\mathbf{I}}_-\otimes\mathbf{f}^{-r}\big)\Big\rangle\\
&T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle = \frac{\tilde{h}(x,v)}{h(x,v)}\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle T^{-n}_{-n}(x)+\\
&\qquad\qquad +\tilde{h}(x,v)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(-)}_{1}(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\tilde{\mathbf{T}}^{(+)}_0(v)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\big(\tilde{\mathbf{I}}_+\otimes\mathbf{e}_k\big)\Big\rangle\\
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1}(w),\mathbf{f}^{-r}\Big\rangle = \Big\langle \mathbf{b}^{(-)}_{1}(w),\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w)\tilde{\mathbf{T}}^{(+)}_0(x)\big(\tilde{\mathbf{I}}_+\otimes\mathbf{f}^{-r}\big)\Big\rangle-\\
&\qquad\qquad -\tilde{h}(w,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\big(\tilde{\mathbf{I}}_+\otimes\mathbf{f}^{-r}\big)\Big\rangle T^{-n}_{-n}(w)\\
&\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v),\mathbf{e}_k\Big\rangle = \Big\langle \mathbf{b}^{(+)}_{1^*}(v),\tilde{\mathbf{T}}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v)\big(\tilde{\mathbf{I}}_-\otimes\mathbf{e}_k\big)\Big\rangle-\\
&\qquad\qquad -\tilde{h}(x,v)\Big\langle \mathbf{b}^{(-)}_{1}(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-1_+}\big(\tilde{\mathbf{I}}_-\otimes\mathbf{e}_k\big)\Big\rangle T^n_n(v)
\end{aligned}
$$

where

$$
\begin{aligned}
&\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+} = \tilde{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,x) = \sum_{i,k=1}^{n-1}\mathbf{E}^i_k\otimes\mathbf{E}^k_i\,, \qquad &&\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-} = \sum_{r,s=1}^{n-1}\mathbf{E}^r_s\otimes\mathbf{F}^{-r}_{-s}\\
&\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-} = \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(w,w) = \sum_{r,s=1}^{n-1}\mathbf{E}^{-r}_{-s}\otimes\mathbf{F}^{-s}_{-r}\,, \qquad &&\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+} = \sum_{i,k=1}^{n-1}\mathbf{E}^{-i}_{-k}\otimes\mathbf{E}^i_k
\end{aligned}
$$

and $\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}$, $\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}$ are the linear mappings $\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}:\mathcal{V}^*_{1^*_-}\to\mathcal{V}_{1_+}$, $\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}:\mathcal{V}_{1_+}\to\mathcal{V}^*_{1^*_-}$ defined by

$$
\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\mathbf{f}^{-r} = \mathbf{e}_r\,, \qquad \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\mathbf{e}_k = \mathbf{f}^{-k}.
$$

Proof: The first two equations are only otherwise written commutation relations

$$
\begin{aligned}
&T^n_n(x)T^n_k(v) = f(v,x)T^n_k(v)T^n_n(x) - g(v,x)T^n_k(x)T^n_n(v)\,,\\
&T^{-n}_{-n}(x)T^{-r}_{-n}(w) = f(x,w)T^{-r}_{-n}(w)T^{-n}_{-n}(x) - g(x,w)T^{-r}_{-n}(x)T^{-n}_{-n}(w)
\end{aligned}
$$

and the third and fourth relationships are the matrix notation of the commutation relations

$$
\begin{aligned}
&T^r_s(x)T^n_k(v) = T^n_k(v)T^r_s(x) + g(x,v)T^n_s(v)T^r_k(x) - g(x,v)T^n_s(x)T^r_k(v)\,,\\
&T^{-i}_{-k}(x)T^{-r}_{-n}(w) = T^{-r}_{-n}(w)T^{-i}_{-k}(x) + g(w,x)T^{-i}_{-n}(w)T^{-r}_{-k}(x) - g(w,x)T^{-i}_{-n}(x)T^{-r}_{-k}(w)\,.
\end{aligned}
$$

To prove the fifth relation, we first use the commutation relation

$$\begin{aligned} T_n^n(x)T_{-n}^{-r}(w) &= T_{-n}^{-r}(w)T_n^n(x) - k(x,w)\sum_{p=1}^{n} T_{-p}^{-r}(w)T_p^n(x) = \\ &= \Big(1-k(x,w)\Big)T_{-n}^{-r}(w)T_n^n(x) - k(x,w)\sum_{p=1}^{n-1} T_{-p}^{-r}(w)T_p^n(x)\,. \end{aligned} \tag{6}$$

If we sum the commutation relations

$$T_{-k}^{-r}(w)T_k^n(x) = T_k^n(x)T_{-k}^{-r}(w) - h(w,x)\sum_{p=1}^{n} T_p^n(x)T_{-p}^{-r}(w)$$

over $k = 1, \ldots, n-1$, we find that

$$\sum_{p=1}^{n-1} T_{-p}^{-r}(w)T_p^n(x) = \Big(1-(n-1)h(w,x)\Big)\sum_{p=1}^{n-1} T_p^n(x)T_{-p}^{-r}(w) - (n-1)h(w,x)T_n^n(x)T_{-n}^{-r}(w)$$

When we substitute this equality into (6), we get

$$T_n^n(x)T_{-n}^{-r}(w) = \Big(1+\tilde{h}(w,x)\Big)T_{-n}^{-r}(w)T_n^n(x) + \tilde{h}(w,x)\sum_{p=1}^{n-1} T_p^n(x)T_{-p}^{-r}(w) \tag{7}$$

which is another notation of the fifth relationship.
To prove the sixth relation, we use the commutation relations

$$T_{-n}^{-n}(x)T_k^n(v) = \Big(1-k(v,x)\Big)T_k^n(v)T_{-n}^{-n}(x) - k(v,x)\sum_{p=1}^{n-1} T_k^p(v)T_{-n}^{-p}(x)\,. \tag{8}$$

If we sum the commutation relations

$$T_k^i(v)T_{-n}^{-i}(x) = T_{-n}^{-i}(x)T_k^i(v) - h(x,v)T_{-n}^{-n}(x)T_k^n(v) - h(x,v)\sum_{p=1}^{n-1} T_{-n}^{-p}(x)T_k^p(v)\,,$$

over $i = 1, \ldots, n-1$, we obtain

$$\sum_{p=1}^{n-1} T_k^p(v)T_{-n}^{-p}(x) = \Big(1-(n-1)h(x,v)\Big)\sum_{p=1}^{n-1} T_{-n}^{-p}(x)T_k^p(v) - (n-1)h(x,v)T_{-n}^{-n}(x)T_k^n(v)\,.$$

When we substitute this relation into (8), we get

$$T_{-n}^{-n}(x)T_k^n(v) = \Big(1+\tilde{h}(x,v)\Big)T_k^n(v)T_{-n}^{-n}(x) + \tilde{h}(x,v)\sum_{p=1}^{n-1} T_{-n}^{-p}(x)T_k^p(v) \tag{9}$$

which can be written in the form shown in Lemma.
To prove the seventh and eighth relationships, we first use the commutation relations

$$\begin{aligned} T_k^i(x)T_{-n}^{-r}(w) &= T_{-n}^{-r}(w)T_k^i(x) - \delta^{i,r}h(w,x)T_{-n}^{-n}(w)T_k^n(x) - \delta^{i,r}h(w,x)\sum_{p=1}^{n-1} T_{-n}^{-p}(w)T_k^p(x) \\ T_{-s}^{-r}(x)T_k^n(v) &= T_k^n(v)T_{-s}^{-r}(x) - \delta_{k,s}h(x,v)T_n^n(v)T_{-n}^{-r}(x) - \delta_{k,s}h(x,v)\sum_{p=1}^{n-1} T_p^n(v)T_{-p}^{-r}(x)\,. \end{aligned}$$

Using equations (9) and (7), we obtain

$$T^i_k(x)T^{-r}_{-n}(w) = T^{-r}_{-n}(w)T^i_k(x) - \delta^{i,r}\tilde{h}(w,x)\sum_{p=1}^{n-1} T^{-p}_{-n}(w)T^p_k(x) - \delta^{i,r}\tilde{h}(w,x)T^n_k(x)T^{-n}_{-n}(w)$$
$$T^{-r}_{-s}(x)T^n_k(v) = T^n_k(v)T^{-r}_{-s}(x) - \delta_{k,s}\tilde{h}(x,v)\sum_{p=1}^{n-1} T^n_p(v)T^{-r}_{-p}(x) - \delta_{k,s}\tilde{h}(x,v)T^{-r}_{-n}(x)T^n_n(v)$$

which are other notations of the last two equations of Lemma. □

Using Lemma 1, it is relatively easy to find members in which x is exchanged with the first component of the vectors $\vec{v}$ and $\vec{w}$. The members, in which x is interchanged with other components of these vectors, can be found by switching the corresponding component to the first place of vectors.
The following Lemma gives a suitable notation of the commutation relations that we use.
Lemma 2. In the RTT–algebra $\mathcal{A}_n$ the following relations apply:

$$\begin{aligned}
&\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{2^*}(y), \mathbf{e}_i\otimes\mathbf{e}_k\Big\rangle = \Big\langle \mathbf{b}^{(+)}_{2^*}(y)\mathbf{b}^{(+)}_{1^*}(x), \widehat{\mathbf{R}}^{(+,+)}_{1_+,2_+}(x,y)\big(\mathbf{e}_i\otimes\mathbf{e}_k\big)\Big\rangle \\
&\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{2}(y), \mathbf{f}^{-r}\otimes\mathbf{f}^{-s}\Big\rangle = \Big\langle \mathbf{b}^{(-)}_{2}(y)\mathbf{b}^{(-)}_{1}(x), \widehat{\mathbf{R}}^{(-,-)}_{1^*_-,2^*_-}(y,x)\big(\mathbf{f}^{-r}\otimes\mathbf{f}^{-s}\big)\Big\rangle \\
&\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{2^*}(y), \mathbf{e}_i\otimes\mathbf{e}_k\Big\rangle = \Big\langle \mathbf{b}^{(+)}_{2^*}(x)\mathbf{b}^{(+)}_{1^*}(y), \widehat{\mathbb{R}}^{(+,+)}_{1_+,2_+}\big(\mathbf{e}_i\otimes\mathbf{e}_k\big)\Big\rangle \\
&\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{2}(y), \mathbf{f}^{-r}\otimes\mathbf{f}^{-s}\Big\rangle = \Big\langle \mathbf{b}^{(-)}_{2}(x)\mathbf{b}^{(-)}_{1}(y), \widehat{\mathbb{R}}^{(-,-)}_{1^*_-,2^*_-}\big(\mathbf{f}^{-r}\otimes\mathbf{f}^{-s}\big)\Big\rangle \\
&\mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2}(y) = \mathbf{b}^{(-)}_{2}(y)\mathbf{b}^{(+)}_{1^*}(x)\,, \\
&\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2}(y), \mathbf{e}_k\otimes\mathbf{f}^{-r}\Big\rangle = \Big\langle \mathbf{b}^{(+)}_{2^*}(x)\mathbf{b}^{(-)}_{1}(y), \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,2_+}\widehat{\mathbb{P}}^{(+,-)}_{2_+,2^*_-}\big(\mathbf{e}_k\otimes\mathbf{f}^{-r}\big)\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(+)}_{2^*}(x)\mathbf{b}^{(-)}_{1}(y), \widehat{\mathbb{P}}^{(+,-)}_{2_+,2^*_-}\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,2^*_-}\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\big(\mathbf{e}_k\otimes\mathbf{f}^{-r}\big)\Big\rangle \\
&\Big\langle \mathbf{b}^{(+)}_{2^*}(x)\mathbf{b}^{(-)}_{1}(y), \mathbf{f}^{-r}\otimes\mathbf{e}_k\Big\rangle = \Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2}(y), \widehat{\mathbb{P}}^{(-,+)}_{2^*_-,2_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,2_+}\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\big(\mathbf{f}^{-r}\otimes\mathbf{e}_k\big)\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2}(y), \widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,2^*_-}\widehat{\mathbb{P}}^{(-,+)}_{2^*_-,2_+}\big(\mathbf{f}^{-r}\otimes\mathbf{e}_k\big)\Big\rangle
\end{aligned}$$

where

$$\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,2^*_-}(y,x) = \frac{1}{f(y,x)}\Big(\tilde{\mathbf{I}}^*_-\otimes\tilde{\mathbf{I}}^*_- + g(y,x)\sum_{r,s=1}^{n-1}\mathbf{F}^{-r}_{-s}\otimes\mathbf{F}^{-s}_{-r}\Big)$$
$$\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,2^*_-} = \widehat{\mathbf{R}}^{(-,-)}_{1^*_-,2^*_-}(x,x) = \sum_{r,s=1}^{n-1}\mathbf{F}^{-r}_{-s}\otimes\mathbf{F}^{-s}_{-r}$$

Proof: The first and second relations are the transcripts of the commutation relations

$$f(x,y)T^n_i(x)T^n_k(y) = T^n_k(y)T^n_i(x) + g(x,y)T^n_i(y)T^n_k(x)$$
$$f(y,x)T^{-r}_{-n}(x)T^{-s}_{-n}(y) = T^{-s}_{-n}(y)T^{-r}_{-n}(x) + g(y,x)T^{-r}_{-n}(y)T^{-s}_{-n}(x)\,,$$

the fifth relation is an otherwise written commutation relation

$$T^n_k(x)T^{-r}_{-n}(y) = T^{-r}_{-n}(y)T^n_k(x)$$

and the other equations are the identities. □

To write the operators' action $T^{\pm n}_{\pm n}(x)$ and $\tilde{\mathbf{T}}^{(\pm)}(x)$ on the Bethe vectors with the general $\vec{v}$ and $\vec{w}$, we prefer to introduce

$$\begin{aligned}
&\mathbf{b}^{(+)}_{k^*;1^*,\ldots,P^*}(x;\vec{v}_k) = \mathbf{b}^{(+)}_{k^*}(x)\mathbf{b}^{(+)}_{1^*,\ldots,\widehat{k}^*,\ldots,P^*}(\vec{v}_k)\\
&\mathbf{b}^{(-)}_{r;1,\ldots,Q}(x;\vec{w}_r) = \mathbf{b}^{(-)}_{r}(x)\mathbf{b}^{(-)}_{1,\ldots,\widehat{r},\ldots,Q}(\vec{w}_r)\\
&\mathbf{b}^{(+)}_{1^*,\ldots,\widehat{k}^*,\ldots,P^*}(\vec{v}_k) = \mathbf{b}^{(+)}_{1^*}(v_1)\ldots\mathbf{b}^{(+)}_{(k-1)^*}(v_{k-1})\mathbf{b}^{(+)}_{(k+1)^*}(v_{k+1})\ldots\mathbf{b}^{(+)}_{P^*}(v_P)\\
&\mathbf{b}^{(-)}_{1,\ldots,\widehat{r},\ldots,Q}(\vec{w}_r) = \mathbf{b}^{(-)}_{1}(w_1)\ldots\mathbf{b}^{(-)}_{r-1}(w_{r-1})\mathbf{b}^{(-)}_{r+1}(w_{r+1})\ldots\mathbf{b}^{(-)}_{Q}(w_Q)\\
&\widehat{\mathbf{R}}^{(+,+)}_{1,\ldots,k}(\vec{v}) = \widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,v_k)\widehat{\mathbf{R}}^{(+,+)}_{2_+,k_+}(v_2,v_k)\ldots\widehat{\mathbf{R}}^{(+,+)}_{(k-1)_+,k_+}(v_{k-1},v_k)\\
&\widehat{\mathbf{R}}^{(-,-)}_{1^*,\ldots,r^*}(\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(w_r,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*_-,r^*_-}(w_r,w_2)\ldots\widehat{\mathbf{R}}^{(-,-)}_{(r-1)^*_-,r^*_-}(w_r,w_{r-1})\\
&\widehat{\mathbf{T}}^{(+,+)}_{0;1,\ldots,P}(x;\vec{v}) = \tilde{\mathbf{T}}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,+)}_{0;1,\ldots,P}(x;\vec{v})\\
&\widehat{\mathbf{T}}^{(-,-)}_{0;1^*,\ldots,Q^*}(x;\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\ldots,Q^*}(x;\vec{w})\tilde{\mathbf{T}}^{(-)}_0(x)\\
&\widehat{\mathbb{T}}^{(+,+)}_{k;0;1,\ldots,P}(\vec{v}) = \widehat{\mathbf{T}}^{(+,+)}_{0;1,\ldots,P}(v_k;\vec{v}) = \tilde{\mathbf{T}}^{(+)}_0(v_k)\widehat{\mathbf{R}}^{(+,+)}_{0;1,\ldots,P}(v_k;\vec{v})\\
&\widehat{\mathbb{T}}^{(-,-)}_{r;0;1^*,\ldots,Q^*}(\vec{w}) = \widehat{\mathbf{T}}^{(-,-)}_{0;1^*,\ldots,Q^*}(w_r;\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\ldots,Q^*}(w_r;\vec{w})\tilde{\mathbf{T}}^{(-)}_0(w_r)\,.
\end{aligned}$$

<u>Lemma 3.</u> For any $\vec{v}$ and $\vec{w}$ the relations

$$\begin{aligned}
&T^n_n(x)\Big\langle\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = F(\overline{v};x)\Big\langle\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(x)-\\
&\qquad -\sum_{v_k\in\overline{v}} g(v_k,x)F(\overline{v}_k;v_k)\Big\langle\mathbf{b}^{(+)}_{k^*;1^*,\ldots,P^*}(x;\vec{v}_k),\widehat{\mathbf{R}}^{(+,+)}_{1,\ldots,k}(\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(v_k)\\
&T^{-n}_{-n}(x)\Big\langle\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = F(x;\overline{w})\Big\langle\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(x)-\\
&\qquad -\sum_{w_r\in\overline{w}} g(x,w_r)F(w_r;\overline{w}_r)\Big\langle\mathbf{b}^{(-)}_{r;1,\ldots,Q}(x;\vec{w}_r),\widehat{\mathbf{R}}^{(-,-)}_{1^*,\ldots,r^*}(\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_r)\\
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = F(x;\overline{v})\Big\langle\mathbf{b}^{(+)}_{1^*,\ldots,P^*}(\vec{v}),\widehat{\mathbf{T}}^{(+,+)}_{0;1,\ldots,P}(x;\vec{v})\,\mathbf{e}_{\vec{k}}\Big\rangle-\\
&\qquad -\sum_{v_k\in\overline{v}} g(x,v_k)F(v_k;\overline{v}_k)\Big\langle\mathbf{b}^{(+)}_{k^*;1^*,\ldots,P^*}(x;\vec{v}_k),\widehat{\mathbf{R}}^{(+,+)}_{1,\ldots,k}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{k;0;1,\ldots,P}(\vec{v})\,\mathbf{e}_{\vec{k}}\Big\rangle\\
&\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = F(\overline{w};x)\Big\langle\mathbf{b}^{(-)}_{1,\ldots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(-,-)}_{0;1^*,\ldots,Q^*}(x;\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\qquad -\sum_{w_r\in\overline{w}} g(w_r,x)F(\overline{w}_r;w_r)\Big\langle\mathbf{b}^{(-)}_{r;1,\ldots,Q}(x;\vec{w}_r),\widehat{\mathbf{R}}^{(-,-)}_{1^*,\ldots,r^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{r;0;1^*,\ldots,Q^*}(\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle
\end{aligned}$$

hold in the RTT–algebra $\mathcal{A}_n$

Proof: We can prove these statements by induction over the number of elements P and Q of the sets $\overline{v}$ and $\overline{w}$.

For $P=Q=1$ these relations are proven in Lemma 1.

We will assume that the statement is valid for P and Q and denote $\vec{v}=(v_1,\ldots,v_P,v_{P+1})$, $\vec{w}=(w_1,\ldots,w_Q,w_{Q+1})$, $\vec{k}=(k_1,\ldots,k_P,k_{P+1})$ and $\vec{r}=(r_1,\ldots,r_Q,r_{Q+1})$. According to the induction assumption and Lemma 1, we have

$$
\begin{aligned}
&T_n^n(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = T_n^n(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1),\mathbf{e}_{k_1}\Big\rangle\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1),\mathbf{e}_{\vec{k}_1}\Big\rangle = \\
&\quad = f(v_1,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1),\mathbf{e}_{k_1}\Big\rangle T_n^n(x)\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1),\mathbf{e}_{\vec{k}_1}\Big\rangle - \\
&\qquad\quad -g(v_1,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\mathbf{e}_{k_1}\Big\rangle T_n^n(v_1)\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1),\mathbf{e}_{\vec{k}_1}\Big\rangle = \\
&\quad = F(\overline{v};x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle T_n^n(x) - \\
&\qquad\quad -g(v_1,x)F(\overline{v}_1;v_1)\Big\langle \mathbf{b}^{(+)}_{1^*;1^*,\dots,(P+1)^*}(x,\vec{v}_1),\mathbf{e}_{\vec{k}}\Big\rangle T_n^n(v_1) - \\
&\quad -\sum_{v_k\in\overline{v}_1} F(\overline{v}_{1,k};v_k)\Big[g(v_k,x)f(v_1,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(x;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle - \\
&\qquad\qquad -g(v_k,v_1)g(v_1,x)\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{k^*;2,\dots,(P+1)^*}(v_1;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle\Big] T_n^n(v_k)\,,
\end{aligned}
$$

$$
\begin{aligned}
&T_{-n}^{-n}(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = T_{-n}^{-n}(x)\Big\langle \mathbf{b}^{(-)}_{1}(w_1),\mathbf{f}^{-r_1}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{2,\dots,Q}(\vec{w}_1),\mathbf{f}^{-\vec{r}_1}\Big\rangle = \\
&\quad = F(x;\overline{w})\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle T_{-n}^{-n}(x) - \\
&\qquad\quad -g(x,w_1)F(w_1;\overline{w}_1)\Big\langle \mathbf{b}^{(-)}_{1;1,\dots,Q}(x,\vec{w}_1),\mathbf{f}^{-\vec{r}}\Big\rangle T_{-n}^{-n}(w_1) - \\
&\quad -\sum_{w_r\in\overline{w}_1} F(w_r;\overline{w}_{1,r})\Big[g(x,w_r)f(x,w_1)\Big\langle \mathbf{b}^{(-)}_{1}(w_1)\mathbf{b}^{(-)}_{r;2,\dots,Q}(x;\vec{w}_{1,r}),\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&\qquad\quad -g(x,w_1)g(w_1,w_r)\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{r;2,\dots,Q}(w_1;\vec{w}_{1,r}),\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle\Big] T_{-n}^{-n}(w_r)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = \tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1)\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1),\mathbf{e}_{\vec{k}_1}\Big\rangle,\mathbf{e}_{k_1}\Big\rangle = \\
&\quad = F(x;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}_1),\widehat{\mathbf{T}}^{(+,+)}_{0;1,\dots,P+1}(x;\vec{v}_1)\,\mathbf{e}_{\vec{k}}\Big\rangle - \\
&\qquad -g(x,v_1)F(v_1;\overline{v}_1)\Big\langle \mathbf{b}^{(+)}_{1^*;1^*,\dots,(P+1)^*}(x;\vec{v}_1),\widehat{\mathbb{T}}^{(+,+)}_{1;0;1,\dots,P+1}(\vec{v})\,\mathbf{e}_{\vec{k}}\Big\rangle - \\
&\qquad -\sum_{v_k\in\overline{v}_1} g(x,v_k)f(x,v_1)F(v_k;\overline{v}_{1,k})\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(x;\vec{v}_{1,k}), \\
&\qquad\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\,\mathbf{e}_{\vec{k}}\Big\rangle + \\
&\qquad +\sum_{v_k\in\overline{v}_1} g(x,v_1)g(v_1,v_k)F(v_k;\overline{v}_{1,k})\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(v_1;\vec{v}_{1,k}), \\
&\qquad\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec{k}}\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = F(\overline{w};x)\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1}(w_1),\mathbf{f}^{-r_1}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1),\mathbf{f}^{-\vec{r}_1}\Big\rangle = \\
&\quad = \Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\widehat{\mathbf{T}}^{(-,-)}_{0;1^*,\dots,(Q+1)^*}(x;\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&\qquad\quad -g(w_1,x)F(\overline{w}_1,w_1)\Big\langle \mathbf{b}^{(-)}_{1;1,\dots,Q+1}(x;\vec{w}_1),\widehat{\mathbb{T}}^{(-,-)}_{1;0;1^*,\dots,(Q+1)^*}(\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&\qquad\quad -\sum_{w_r\in\overline{w}_1} g(w_r,x)f(w_1,x)F(\overline{w}_{1,r},w_r)\Big\langle \mathbf{b}^{(-)}_{1}(w_1)\mathbf{b}^{(-)}_{r;2,\dots,Q+1}(x;\vec{w}_{1,r}), \\
&\qquad\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle + \\
&\qquad\quad +\sum_{w_r\in\overline{w}_1} g(w_r,w_1)g(w_1,x)F(\overline{w}_{1,r},w_r)\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{r;2,\dots,Q+1}(w_1;\vec{w}_{1,r}), \\
&\qquad\qquad\qquad \widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle.
\end{aligned}
$$

If we use in the first two equations the relations

$$
\begin{aligned}
&\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(x;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle \\
&\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(v_1;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\widehat{\mathbb{R}}^{(+,+)}_{1_+,k_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle \\
&\Big\langle \mathbf{b}^{(-)}_{1}(w_1)\mathbf{b}^{(-)}_{r;2,\dots,Q}(x;\vec{w}_{1,r}),\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q}(x;\vec{w}_r),\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle \\
&\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{r;2,\dots,Q}(w_1;\vec{w}_{1,r}),\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\qquad\qquad = \Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q}(x;\vec{w}_r),\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle
\end{aligned}
$$

that result from Lemma 2, and compare the results with the first two relations of the proven Lemma, we can see that it is enough to show for any $v_k \in \overline{v}_1$ and any $w_r \in \overline{w}_1$ the equalities

$$
\begin{aligned}
&g(v_k,x)f(v_1,v_k)\widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,v_k) = g(v_k,x)f(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,x) - g(v_k,v_1)g(v_1,x)\widehat{\mathbb{R}}^{(+,+)}_{1_+,k_+}\,, \\
&g(x,w_r)f(w_r,w_1)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(w_r,w_1) = \\
&\qquad\qquad = g(x,w_r)f(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1) - g(x,w_1)g(w_1,w_r)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\,.
\end{aligned}
$$

However, this is equivalent to the identities

$$
\begin{aligned}
g(v_k,x)g(v_1,v_k) &= g(v_k,x)g(v_1,x) - g(v_k,v_1)g(v_1,x) \\
g(x,w_r)g(w_r,w_1) &= g(x,w_r)g(x,w_1) - g(x,w_1)g(w_1,w_r)\,.
\end{aligned}
$$

To prove the third and fourth equality of Lemma, we use the relations

$$
\begin{aligned}
&\Big\langle \mathbf{b}^{(+)}_{1^*}(v_1)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(x;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\,\mathbf{e}_{\vec{k}}\Big\rangle = \\
&\quad = \Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\,\mathbf{e}_{\vec{k}}\Big\rangle \\
&\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(+)}_{k^*;2^*,\dots,(P+1)^*}(v_1;\vec{v}_{1,k}),\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec{k}}\Big\rangle = \\
&\quad = \Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\widehat{\mathbb{R}}^{(+,+)}_{1,k}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec{k}}\Big\rangle, \\
&\Big\langle \mathbf{b}^{(-)}_{1}(w_1)\mathbf{b}^{(-)}_{r;2,\dots,Q+1}(x;\vec{w}_{1,r}),\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\quad = \Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q+1}(x;\vec{w}_r),\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle \\
&\Big\langle \mathbf{b}^{(-)}_{1}(x)\mathbf{b}^{(-)}_{r;2,\dots,Q+1}(w_1;\vec{w}_{1,r}),\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\quad = \Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q+1}(x;\vec{w}_r),\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle,
\end{aligned}
$$

that follow from Lemma 2. Then we get the equalities

$$
\begin{aligned}
&\tilde{\mathbf{T}}_0^{(+)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = F(x,\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}_1),\widehat{\mathbf{T}}^{(+,+)}_{0;1,\dots,P+1}(x;\vec{v}_1)\,\mathbf{e}_{\vec{k}}\Big\rangle-\\
&\quad -g(x,v_1)F(v_1,\overline{v}_1)\Big\langle \mathbf{b}^{(+)}_{1^*;1^*,\dots,(P+1)^*}(x;\vec{v}_1),\widehat{\mathbb{T}}^{(+,+)}_{1;0;1,\dots,P+1}(\vec{v})\,\mathbf{e}_{\vec{k}}\Big\rangle-\\
&\quad -\sum_{v_k\in\overline{v}_1}F(v_k,\overline{v}_{1,k})\Big(g(x,v_k)f(x,v_1)\Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\\
&\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)\,\mathbf{e}_{\vec{k}}\Big\rangle-\\
&\qquad -g(x,v_1)g(v_1,v_k)\Big\langle \mathbf{b}^{(+)}_{k^*;1^*,\dots,(P+1)^*}(x;\vec{v}_k),\\
&\qquad\qquad \widehat{\mathbb{R}}^{(+,+)}_{1,k}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec{k}}\Big\rangle\Big)\\
&\tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = F(\overline{w};x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\widehat{\mathbf{T}}^{(-,-)}_{0;1^*,\dots,(Q+1)^*}(x;\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\quad -g(w_1,x)F(\overline{w}_1,w_1)\Big\langle \mathbf{b}^{(-)}_{1;1,\dots,Q+1}(x;\vec{w}_1),\widehat{\mathbb{T}}^{(-,-)}_{1;0;1^*,\dots,(Q+1)^*}(\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\quad -\sum_{w_r\in\overline{w}_1}F(\overline{w}_{1,r},w_r)\Big(g(w_r,x)f(w_1,x)\Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q+1}(x;\vec{w}_r),\\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\qquad -g(w_r,w_1)g(w_1,x)\Big\langle \mathbf{b}^{(-)}_{r;1,\dots,Q+1}(x;\vec{w}_r),\\
&\qquad\qquad \widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle\Big).
\end{aligned}
$$

If we show that the relations

$$
\begin{aligned}
&g(x,v_k)f(v_k,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,k}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{k;0;1,\dots,P+1}(\vec{v}) =\\
&\quad = g(x,v_k)f(x,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)-\\
&\qquad -g(x,v_1)g(v_1,v_k)\widehat{\mathbb{R}}^{(+,+)}_{1,k}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+},\\
&g(w_r,x)f(w_1,w_r)\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,r^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{r;0;1^*,\dots,(Q+1)^*}(\vec{w}) =\\
&\quad = g(w_r,x)f(w_1,x)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)-\\
&\qquad -g(w_r,w_1)g(w_1,x)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1),
\end{aligned}
$$

are valid for any $v_k\in\overline{v}_1$ and $w_r\in\overline{w}_1$, the third and fourth equality in Lemma will hold. Since, by definition

$$
\begin{aligned}
&\widehat{\mathbb{T}}^{(+,+)}_{k;0;1,\dots,P+1}(\vec{v}) = \tilde{\mathbf{T}}_0^{(+)}(v_k)\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P+1}(v_k;\vec{v})\\
&\widehat{\mathbb{T}}^{(+,+)}_{k;0;2,\dots,P+1}(\vec{v}_1) = \tilde{\mathbf{T}}_0^{(+)}(v_k)\widehat{\mathbf{R}}^{(+,+)}_{0;2,\dots,P+1}(v_k;\vec{v}_1)\\
&\widehat{\mathbb{T}}^{(-,-)}_{r;0;1^*,\dots,(Q+1)^*}(\vec{w}) = \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,(Q+1)^*}(w_r;\vec{w})\tilde{\mathbf{T}}_0^{(-)}(w_r)\\
&\widehat{\mathbb{T}}^{(-,-)}_{r;0;2^*,\dots,(Q+1)^*}(\vec{w}_1) = \widehat{\mathbf{R}}^{(-,-)}_{0;2^*,\dots,(Q+1)^*}(w_r;\vec{w}_1)\tilde{\mathbf{T}}_0^{(-)}(w_r)
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&g(x,v_k)f(v_k,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,k}(\vec{v})\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P+1}(v_k;\vec{v}) = \\
&\quad = g(x,v_k)f(x,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0;2,\dots,P+1}(v_k;\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)- \\
&\qquad -g(x,v_1)g(v_1,v_k)\widehat{\mathbb{R}}^{(+,+)}_{1,k}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,k}(\vec{v}_1)\widehat{\mathbf{R}}^{(+,+)}_{0;2,\dots,P+1}(v_k;\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+} \\
&g(w_r,x)f(w_1,w_r)\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,r^*}(\vec{w})\widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,(Q+1)^*}(w_r;\vec{w}) = \\
&\quad = g(w_r,x)f(w_1,x)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbf{R}}^{(-,-)}_{0;2^*,\dots,(Q+1)^*}(w_r;\vec{w}_1)- \\
&\qquad -g(w_r,w_1)g(w_1,x)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,r^*}(\vec{w}_1)\widehat{\mathbf{R}}^{(-,-)}_{0;2^*,\dots,(Q+1)^*}(w_r;\vec{w}_1)
\end{aligned}
$$

are true.
If we use the definitions of the products of R–matrices, we find that it is enough to show

$$
\begin{aligned}
&g(x,v_k)f(v_k,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,v_k)\dots\widehat{\mathbf{R}}^{(+,+)}_{k-1,k}(v_{k-1},v_k) \\
&\qquad\qquad \widehat{\mathbb{R}}^{(+,+)}_{0,k}\widehat{\mathbf{R}}^{(+,+)}_{0,k-1}(v_k,v_{k-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0,1}(v_k,v_1) = \\
&\quad = g(x,v_k)f(x,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,k}(v_1,x)\widehat{\mathbf{R}}^{(+,+)}_{2,k}(v_2,v_k)\dots\widehat{\mathbf{R}}^{(+,+)}_{k-1,k}(v_{k-1},v_k) \\
&\qquad\qquad \widehat{\mathbb{R}}^{(+,+)}_{0,k}\widehat{\mathbf{R}}^{(+,+)}_{0,k-1}(v_k,v_{k-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0,2}(v_k,v_2)\widehat{\mathbf{R}}^{(+,+)}_{0,1}(x,v_1)- \\
&\qquad -g(x,v_1)g(v_1,v_k)\widehat{\mathbb{R}}^{(+,+)}_{1,k}\widehat{\mathbf{R}}^{(+,+)}_{2,k}(v_2,v_k)\dots\widehat{\mathbf{R}}^{(+,+)}_{k-1,k}(v_{k-1},v_k) \\
&\qquad\qquad \widehat{\mathbb{R}}^{(+,+)}_{0,k}\widehat{\mathbf{R}}^{(+,+)}_{0,k-1}(v_k,v_{k-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0,2}(v_k,v_2)\widehat{\mathbb{R}}^{(+,+)}_{0,1}, \\
&g(w_r,x)f(w_1,w_r)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(w_r,w_1)\dots\widehat{\mathbf{R}}^{(-,-)}_{(r-1)^*_-,r^*_-}(w_r,w_{r-1}) \\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(w_r,w_1)\dots\widehat{\mathbf{R}}^{(-,-)}_{0_-,(r-1)^*_-}(w_r,w_{r-1})\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-} = \\
&\quad = g(w_r,x)f(w_1,x)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{2^*_-,r^*_-}(w_r,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{(r-1)^*_-,r^*_-}(w_r,w_{r-1}) \\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(w_r,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{0_-,(r-1)^*_-}(w_r,w_{r-1})\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-}- \\
&\qquad -g(w_r,w_1)g(w_1,x)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*_-,r^*_-}(w_r,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{(r-1)^*_-,r^*_-}(w_r,w_{r-1}) \\
&\qquad\qquad \widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(w_r,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{0_-,(r-1)^*_-}(w_r,w_{r-1})\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-}.
\end{aligned}
$$

When we use the Yang–Baxter equations

$$
\begin{aligned}
\widehat{\mathbf{R}}^{(+,+)}_{r_+,k_+}(v_r,v_k)\widehat{\mathbb{R}}^{(+)}_{0_+,k_+}\widehat{\mathbf{R}}^{(+,+)}_{0_+,r_+}(v_k,v_r) &= \widehat{\mathbf{R}}^{(+,+)}_{0_+,r_+}(v_k,v_r)\widehat{\mathbb{R}}^{(+)}_{0_+,k_+}\widehat{\mathbf{R}}^{(+,+)}_{r_+,k_+}(v_r,v_k) \\
\widehat{\mathbf{R}}^{(-,-)}_{s^*_-,r^*_-}(w_r,w_s)\widehat{\mathbf{R}}^{(-,-)}_{0_-,s^*_-}(w_r,w_s)\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-} &= \widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-}\widehat{\mathbf{R}}^{(-,-)}_{0_-,s^*_-}(w_r,w_s)\widehat{\mathbf{R}}^{(-,-)}_{s^*_-,r^*_-}(w_r,w_s)
\end{aligned}
$$

several times, it can be found that it is enough to prove the relations

$$
\begin{aligned}
&g(x,v_k)f(v_k,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,v_k)\widehat{\mathbb{R}}^{(+,+)}_{0_+,k_+}\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(v_k,v_1) = \\
&\quad = g(x,v_k)f(x,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1_+,k_+}(v_1,x)\widehat{\mathbb{R}}^{(+,+)}_{0_+,k_+}\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(x,v_1)- \\
&\qquad -g(x,v_1)g(v_1,v_k)\widehat{\mathbb{R}}^{(+,+)}_{1_+,k_+}\widehat{\mathbb{R}}^{(+,+)}_{0_+,k_+}\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}, \\
&g(w_r,x)f(w_1,w_r)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(w_r,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(w_r,w_1)\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-} = \\
&\quad = g(w_r,x)f(w_1,x)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,r^*_-}(x,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(x,w_1)\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-}- \\
&\qquad -g(w_r,w_1)g(w_1,x)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,r^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,r^*_-},
\end{aligned}
$$

which can be verified by direct calculation. $\square$

Lemma 4. For any $\vec{v}$, $\vec{w}$ and $\vec{k}$, $\vec{r}$ the following relations are true:

$$
\begin{aligned}
T^n_n(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle &= F(\overline{w};x-n+1+\eta)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(x)+\\
&+\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(\overline{w}_s;w_s)\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{s;0;1^*,\dots,Q^*}(\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle\bigg)\\
T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle &= F(x+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle T^{-n}_{-n}(x)+\\
&+\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(v_\ell;\overline{v}_\ell)\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{\ell}(x),\\
&\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P}(\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle\bigg)\\
\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle &= \Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&-\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(w_s;\overline{w}_s)\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)\\
\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle &= \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\widehat{\mathbf{T}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle-\\
&-\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(\overline{v}_\ell;v_\ell)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{\ell}(x),\\
&\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(v_\ell)
\end{aligned}
$$

where

$$
\widehat{\mathbf{T}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})=\widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\tilde{\mathbf{T}}^{(+)}_0(x)\,,\qquad \widehat{\mathbf{T}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v})=\tilde{\mathbf{T}}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v})\,.
$$

Proof: These statements can be proven by induction according to the number of elements P and Q of the sets $\overline{v}$ and $\overline{w}$. For $P=1$ and $Q=1$, these statements are proved in Lemma 3.

Assume that these statements hold for P and Q and denote $\vec{v}=(v_1,\dots,v_{P+1})$, $\vec{w}=(w_1,\dots,w_{Q+1})$, $\vec{k}=(k_1,\dots,k_{P+1})$ and $\vec{r}=(r_1,\dots,r_{Q+1})$.

To show the first statement, we use the equality

$$
\begin{aligned}
T^n_n(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle &= T^n_n(x)\Big\langle \mathbf{b}^{(-)}_{1}(w_1),\mathbf{f}^{-r_1}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1),\mathbf{f}^{-\vec{r}_1}\Big\rangle=\\
&=\frac{\tilde{h}(w_1,x)}{h(w_1,x)}\Big\langle \mathbf{b}^{(-)}_{1}(w_1),\mathbf{f}^{-r_1}\Big\rangle T^n_n(x)\Big\langle \mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1),\mathbf{f}^{-\vec{r}_1}\Big\rangle+\\
&\quad+\tilde{h}(w_1,x)\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*}(x),\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\mathbf{f}^{-r_1}\Big\rangle\tilde{\mathbf{T}}^{(-)}_0(w_1)\Big\langle \mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1),\mathbf{f}^{-\vec{r}_1}\Big\rangle\bigg),
\end{aligned}
$$

which results from Lemma 1. Using the induction assumption and Lemma 3, we will get

$$T_n^n(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = F(\overline{w};x-n+1+\eta)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle T_n^n(x)+$$
$$+\tilde{h}(w_1,x)F(\overline{w}_1;w_1)\operatorname{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1),$$
$$\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbb{T}}^{(-,-)}_{1;0;1^*,\dots,(Q+1)^*}(\vec{w})\,\mathbf{f}^{-\vec{r}}\Big\rangle\bigg)+$$
$$+\sum_{w_s\in\overline{w}_1}\frac{\tilde{h}(w_1,x)}{h(w_1,x)}\tilde{h}(w_s,x)F(\overline{w}_{1,s};w_s)\operatorname{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q+1}(\vec{w}_s),$$
$$\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle\bigg)-$$
$$-\sum_{w_s\in\overline{w}_1}\tilde{h}(w_1,x)g(w_s,w_1)F(\overline{w}_{1,s};w_s)\operatorname{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{s;2,\dots,Q+1}(w_1;\vec{w}_{1,s}),$$
$$\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle\bigg).$$

When we use Lemma 2 and the relationship $\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}=\tilde{\mathbf{I}}_{1^*_-}$, we obtain the relation

$$\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{s;2,\dots,Q+1}(w_1;\vec{w}_{1,s}),\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle=$$
$$=\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q+1}(\vec{w}_s),\mathbb{P}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,\mathbf{f}^{-\vec{r}}\Big\rangle.$$

So it is enough to show that for any $s=2,\dots,Q+1$ we have

$$\tilde{h}(w_s,x)f(w_1,w_s)\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{s;0;1^*,\dots,(Q+1)^*}(\vec{w})=$$
$$=\frac{\tilde{h}(w_1,x)}{h(w_1,x)}\tilde{h}(w_s,x)\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)-$$
$$-\tilde{h}(w_1,x)g(w_s,w_1)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\widehat{\mathbb{T}}^{(-,-)}_{s;0;2^*,\dots,(Q+1)^*}(\vec{w}_1)\,.$$

It follows from the definitions of the operators that in order to prove a statement, it is sufficient to prove the relation

$$\tilde{h}(w_s,x)f(w_1,w_s)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,s^*_-}(w_s,w_1)\dots\widehat{\mathbf{R}}^{(-,-)}_{(s-1)^*_-,s^*_-}(w_s,w_{s-1})\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(w_s,w_1)\dots\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}=$$
$$=\frac{\tilde{h}(w_1,x)}{h(w_1,x)}\tilde{h}(w_s,x)\widehat{\mathbf{R}}^{(-,-)}_{2^*_-,s^*_-}(w_s,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{(s-1)^*_-,s^*_-}(w_s,w_{s-1})$$
$$\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(w_s,w_2)\dots\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}-$$
$$-\tilde{h}(w_1,x)g(w_s,w_1)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*_-,s^*_-}(w_s,w_2)\dots\widehat{\mathbf{R}}^{(-,-)}_{(s-1)^*_-,s^*_-}(w_s,w_{s-1})$$
$$\widehat{\mathbf{R}}^{(-,-)}_{0_-,2^*_-}(w_s,w_2)\dots\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}$$

By direct calculation it is possible to show that the Yang–Baxter equation

$$\widehat{\mathbf{R}}^{(-,-)}_{t^*_-,s^*_-}(w_s,w_t)\widehat{\mathbf{R}}^{(-,-)}_{0_-,t^*_-}(w_s,w_t)\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}=\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{0_-,t^*_-}(w_s,w_t)\widehat{\mathbf{R}}^{(-,-)}_{t^*_-,s^*_-}(w_s,w_t)$$

holds and by its repeated use we find that for the proof of the first statement it is sufficient to prove the relation

$$\tilde h(w_s,x) f(w_1,w_s)\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,s^*_-}(w_s,w_1)\widehat{\mathbf{R}}^{(-,-)}_{0_-,1^*_-}(w_s,w_1)\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-} = \\ = \frac{\tilde h(w_1,x)}{h(w_1,x)}\tilde h(w_s,x)\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-} - \tilde h(w_1,x) g(w_s,w_1)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,1^*_-}\widehat{\mathbb{R}}^{(-,-)}_{0_-,s^*_-}\,.$$

But this relationship is equivalent to the identity

$$\tilde h(w_s,x) f(w_1,w_s) = \frac{\tilde h(w_1,x)}{h(w_1,x)}\tilde h(w_s,x) - \tilde h(w_1,x) g(w_s,w_1)\,.$$

To prove the second relationship, we use Lemmas 1 and 3, from which it follows

$$T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec v),\mathbf{e}_{\vec k}\Big\rangle = T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*},\mathbf{e}_{k_1}\Big\rangle\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec v_1),\mathbf{e}_{\vec k_1}\Big\rangle = \\ = F(x+n-1-\eta;\overline v)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec v_1),\mathbf{e}_{\vec k}\Big\rangle T^{-n}_{-n}(x) + \\ +\tilde h(x,v_1)F(v_1;\overline v_1)\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec v_1)\mathbf{b}^{(-)}_1(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{T}}^{(+,+)}_{1;0;2,\dots,P+1}(\vec v)\mathbf{e}_{\vec k}\Big\rangle\bigg) + \\ + \sum_{v_\ell\in\overline v_1}\frac{\tilde h(x,v_1)}{h(x,v_1)}\tilde h(x,v_\ell)F(v_\ell;\overline v_{1,\ell})\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,(P+1)^*}(\vec v_\ell)\mathbf{b}^{(-)}_\ell(x), \\ \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\mathbf{e}_{\vec k}\Big\rangle\bigg) - \\ - \sum_{v_\ell\in\overline v_1}\tilde h(x,v_1)g(v_1,v_\ell)F(v_\ell;\overline v_{1,\ell})\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{\ell^*;2^*,\dots,(P+1)^*}(v_1;\vec v_{1,\ell})\mathbf{b}^{(-)}_1(x), \\ \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec k}\Big\rangle\bigg)$$

According to Lemma 2,

$$\Big\langle \mathbf{b}^{(+)}_{\ell^*;2^*,\dots,(P+1)^*}(v_1;\vec v_{1,\ell})\mathbf{b}^{(-)}_1(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec k}\Big\rangle = \\ = \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,(P+1)^*}(\vec v_\ell)\mathbf{b}^{(-)}_\ell(x),\widehat{\mathbb{P}}^{(-,+)}_{2\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec k}\Big\rangle,$$

and so

$$T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec v),\mathbf{e}_{\vec k}\Big\rangle = F(x+n-1-\eta;\overline v)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec v_1),\mathbf{e}_{\vec k}\Big\rangle T^{-n}_{-n}(x) + \\ +\tilde h(x,v_1)F(v_1;\overline v_1)\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec v_1)\mathbf{b}^{(-)}_1(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{T}}^{(+,+)}_{1;0;2,\dots,P+1}(\vec v)\mathbf{e}_{\vec k}\Big\rangle\bigg) + \\ + \sum_{v_\ell\in\overline v_1}\frac{\tilde h(x,v_1)}{h(x,v_1)}\tilde h(x,v_\ell)F(v_\ell;\overline v_{1,\ell})\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,(P+1)^*}(\vec v_\ell)\mathbf{b}^{(-)}_\ell(x), \\ \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\mathbf{e}_{\vec k}\Big\rangle\bigg) - \\ - \sum_{v_\ell\in\overline v_1}\tilde h(x,v_1)g(v_1,v_\ell)F(v_\ell;\overline v_{1,\ell})\,\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,(P+1)^*}(\vec v_\ell)\mathbf{b}^{(-)}_\ell(x), \\ \widehat{\mathbb{P}}^{(-,+)}_{2\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec v_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec v_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}\,\mathbf{e}_{\vec k}\Big\rangle\bigg)$$

Therefore, it is sufficient to show that the relation

$$\tilde{h}(x,v_\ell)f(v_\ell,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P+1}(\vec{v}) =$$

$$= \frac{\tilde{h}(x,v_1)}{h(x,v_1)}\tilde{h}(x,v_\ell)\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec{v}_1) -$$

$$-\tilde{h}(x,v_1)g(v_1,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec{v}_1)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}$$

is valid for any $\ell = 2, \dots, P+1$.
When we use the definitions $\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})$, $\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v})$, $\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P+1}(\vec{v})$ and $\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;2,\dots,P+1}(\vec{v}_1)$, we find that to prove the second statement, it is enough to show the equality

$$\tilde{h}(x,v_\ell)f(v_\ell,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,\ell}(v_1,v_\ell)\dots\widehat{\mathbf{R}}^{(+,+)}_{\ell-1,\ell}(v_{\ell-1},v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}$$

$$\widehat{\mathbf{R}}^{(+,+)}_{0_+,(\ell-1)_+}(v_\ell,v_{\ell-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(v_\ell,v_1) =$$

$$= \frac{\tilde{h}(x,v_1)}{h(x,v_1)}\tilde{h}(x,v_\ell)\widehat{\mathbf{R}}^{(+,+)}_{2,\ell}(v_2,v_\ell)\dots\widehat{\mathbf{R}}^{(+,+)}_{\ell-1,\ell}(v_{\ell-1},v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}$$

$$\widehat{\mathbf{R}}^{(+,+)}_{0_+,(\ell-1)_+}(v_\ell,v_{\ell-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0_+,2_+}(v_\ell,v_2) -$$

$$-\tilde{h}(x,v_1)g(v_1,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\ell}(v_2,v_\ell)\dots\widehat{\mathbf{R}}^{(+,+)}_{\ell-1,\ell}(v_{\ell-1},v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}$$

$$\widehat{\mathbf{R}}^{(+,+)}_{0_+,(\ell-1)_+}(v_\ell,v_{\ell-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{0_+,2_+}(v_\ell,v_2)\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}.$$

By repeatedly using the Yang–Baxter equation

$$\widehat{\mathbf{R}}^{(+,+)}_{k_+,\ell_+}(v_k,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{0_+,k_+}(v_\ell,v_k) = \widehat{\mathbf{R}}^{(+,+)}_{0_+,k_+}(v_\ell,v_k)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{k_+,\ell_+}(v_k,v_\ell),$$

which can be verified by direct calculation, we find that to prove the statement it is enough to prove the relation

$$\tilde{h}(x,v_\ell)f(v_\ell,v_1)\widehat{\mathbf{R}}^{(+,+)}_{1,\ell}(v_1,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{0_+,1_+}(v_\ell,v_1) =$$

$$= \frac{\tilde{h}(x,v_1)}{h(x,v_1)}\tilde{h}(x,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+} - \tilde{h}(x,v_1)g(v_1,v_\ell)\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{0_+,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{0_+,1_+}$$

which is equivalent to the identity

$$\tilde{h}(x,v_\ell)f(v_\ell,v_1) = \frac{\tilde{h}(x,v_1)}{h(x,v_1)}\tilde{h}(x,v_\ell) - \tilde{h}(x,v_1)g(v_1,v_\ell).$$

Assuming that the third statement holds for Q, we get by Lemmas 1 and 3

$$\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}), \mathbf{f}^{-\vec{r}}\Big\rangle = \Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}), \widehat{\mathbf{T}}^{(+,-)}_{0;1,\dots,Q+1}(x;\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle -$$

$$-\tilde{h}(w_1,x)F(w_1;\overline{w}_1)\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1), \widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_1) -$$

$$-\sum_{w_s\in\overline{w}_1}\tilde{h}(w_s,x)F(w_s;\overline{w}_{1,s})\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q+1}(\vec{w}_s),$$

$$\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w_1)\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s) +$$

$$+\sum_{w_s\in\overline{w}_1}\tilde{h}(w_1,x)g(w_1,w_s)F(w_s;\overline{w}_{1,s})\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{s;2,\dots,Q+1}(w_1;\vec{w}_{1,s}),$$

$$\widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)$$

According to Lemma 2,

$$\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{s;2,\dots,Q+1}(w_1;\vec{w}_{1,s}), \widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle =$$
$$= \Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q+1}(w_1;\vec{w}_s), \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle$$

and so

$$\begin{aligned}
\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}), \mathbf{f}^{-\vec{r}}\Big\rangle &= \Big\langle \mathbf{b}^{(-)}_{1,\dots,Q+1}(\vec{w}), \widehat{\mathbf{T}}^{(+,-)}_{0;1,\dots,Q+1}(x;\vec{w})\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&-\tilde{h}(w_1,x)F(w_1;\overline{w}_1)\Big\langle \mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_{2,\dots,Q+1}(\vec{w}_1), \widehat{\mathbb{P}}^{(+,-)}_{1_+,1^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_1) - \\
&- \sum_{w_s\in\overline{w}_1}\tilde{h}(w_s,x)F(w_s;\overline{w}_{1,s})\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q+1}(\vec{w}_s), \\
&\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w_1)\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s) + \\
&+ \sum_{w_s\in\overline{w}_1}\tilde{h}(w_1,x)g(w_1,w_s)F(w_s;\overline{w}_{1,s})\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q+1}(w_1;\vec{w}_s), \\
&\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\big(\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\big)^*\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}\widehat{\mathbf{R}}^{(-,-)}_{2^*,\dots,s^*}(\vec{w}_1)\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)
\end{aligned}$$

Therefore, it is enough to show that for any $s = 2,\dots,Q+1$ we have

$$\tilde{h}(w_s,x)f(w_s,w_1)\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*_-,s^*_-}(w_s,w_1) =$$
$$= \tilde{h}(w_s,x)\widehat{\mathbf{R}}^{(+,-)}_{0_+,1^*_-}(x,w_1)\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-} - \tilde{h}(w_1,x)g(w_1,w_s)\widehat{\mathbb{R}}^{(-,-)}_{1^*_-,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,1^*_-}$$

If we use the definitions of these mappings, we find that these relations are equivalent to identity

$$\tilde{h}(w_s,x)g(w_s,w_1) + \tilde{h}(w_s,x)\tilde{h}(w_1,x) + \tilde{h}(w_1,x)g(w_1,w_s) = 0\,.$$

To prove the fourth statement, we use the relation

$$\begin{aligned}
\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}), \mathbf{e}_{\vec{k}}\Big\rangle &= \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}), \widehat{\mathbf{T}}^{(-,+)}_{0;1,\dots,P+1}(x;\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle - \\
&-\tilde{h}(x,v_1)F(\overline{v}_1;v_1)\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1)\mathbf{b}^{(-)}_1(x), \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\mathbf{e}_{\vec{k}}\Big\rangle T^{n}_{n}(v_1) - \\
&- \sum_{v_\ell\in\overline{v}_1}\tilde{h}(x,v_\ell)F(\overline{v}_{1,\ell};v_\ell)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,(P+1)^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_\ell(x), \\
&\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v_1)\mathbf{e}_{\vec{k}}\Big\rangle T^{n}_{n}(v_\ell) + \\
&+ \sum_{v_\ell\in\overline{v}_1}\tilde{h}(x,v_1)g(v_\ell,v_1)F(\overline{v}_{1,\ell};v_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;2^*,\dots,(P+1)^*}(v_1;\vec{v}_{1,\ell})\mathbf{b}^{(-)}_1(x), \\
&\qquad \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle T^{n}_{n}(v_\ell)\,.
\end{aligned}$$

which follows from Lemma 1 and the inductive assumption. According to Lemma 2, we have

$$\Big\langle \mathbf{b}^{(+)}_{\ell^*;2^*,\dots,(P+1)^*}(v_1;\vec{v}_{1,\ell})\mathbf{b}^{(-)}_1(x), \widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle =$$
$$= \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,(P+1)^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_1(x), \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle$$

and so

$$
\begin{aligned}
\tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle &= \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,(P+1)^*}(\vec{v}),\widehat{\mathbf{T}}^{(-,+)}_{0;1,\dots,P+1}(x;\vec{v})\mathbf{e}_{\vec{k}}\Big\rangle - \\
&-\tilde{h}(x,v_1)F(\overline{v}_1;v_1)\Big\langle \mathbf{b}^{(+)}_{2^*,\dots,(P+1)^*}(\vec{v}_1)\mathbf{b}^{(-)}_1(x),\widehat{\mathbb{P}}^{(-,+)}_{1^*_-,1_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(v_1) - \\
&-\sum_{v_\ell\in\overline{v}_1}\tilde{h}(x,v_\ell)F(\overline{v}_{1,\ell};v_\ell)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,(P+1)^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_\ell(x), \\
&\qquad\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v_1)\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(v_\ell) + \\
&+\sum_{v_\ell\in\overline{v}_1}\tilde{h}(x,v_1)g(v_\ell,v_1)F(\overline{v}_{1,\ell};v_\ell)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{l^*},\dots,(P+1)^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_1(x), \\
&\qquad\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\widehat{\mathbf{R}}^{(+,+)}_{2,\dots,\ell}(\vec{v}_1)\mathbf{e}_{\vec{k}}\Big\rangle T^n_n(v_\ell)
\end{aligned}
$$

Therefore, it is enough to show that equality

$$
\begin{aligned}
&\tilde{h}(x,v_\ell)f(v_1,v_\ell)\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\ell}(v_1,v_\ell) = \\
&\quad = \tilde{h}(x,v_\ell)\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(-,+)}_{0_-,1_+}(x,v_1) - \tilde{h}(x,v_1)g(v_\ell,v_1)\widehat{\mathbb{R}}^{(+,+)}_{1_+,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,1_+}\,.
\end{aligned}
$$

holds for any $\ell = 2,\ \dots,\ P+1$. And if we use the definitions of the involved operators, we find that this equality is equivalent to the relation

$$
\tilde{h}(x,v_\ell)g(v_1,w_\ell) + \tilde{h}(x,v_\ell)\tilde{h}(x,v_1) + \tilde{h}(x,v_1)g(v_\ell,v_1) = 0\,,
$$

which can be easily verified. □

Lemma 5. For any $\vec{v}$, $\vec{w}$, $\vec{k}$ and $\vec{r}$ the relations

$$
\begin{aligned}
&T^n_n(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\quad = F(\overline{v};x)F(\overline{w};x-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(x) - \\
&\qquad -\sum_{v_\ell\in\overline{v}} g(v_\ell,x)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \\
&\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(v_\ell) + \\
&\qquad +\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(\overline{w}_s;w_s)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec{w}_s), \\
&\qquad\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(w_s;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle \\
&\tilde{\mathbf{T}}_0^{(+)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\quad = F(x;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&\qquad -\sum_{v_\ell\in\overline{v}} g(x,v_\ell)F(v_\ell;\overline{v}_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \\
&\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(v_\ell;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle - \\
&\qquad -\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s), \\
&\qquad\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)
\end{aligned}
$$

$$
\begin{aligned}
&T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle=\\
&\quad=F(x;\overline{w})F(x+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle T^{-n}_{-n}(x)-\\
&\qquad-\sum_{w_s\in\overline{w}} g(x,w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec w_s),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle T^{-n}_{-n}(w_s)+\\
&\qquad+\sum_{v_\ell\in\overline{v}}\tilde h(x,v_\ell)F(v_\ell;\overline{v}_\ell)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec v_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec v)\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(v_\ell;\vec v;\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle\\
&\tilde{\mathbf{T}}^{(-)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle=\\
&\quad=F(\overline{w};x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec v;\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle-\\
&\qquad-\sum_{w_s\in\overline{w}} g(w_s,x)F(\overline{w}_s;w_s)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec w_s),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec w)\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(w_s;\vec v,\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle-\\
&\qquad-\sum_{v_\ell\in\overline{v}}\tilde h(x,v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec v_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec v)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle T^n_n(v_\ell)
\end{aligned}
$$

hold.

Proof: In proving this Lemma, we will often use the equation

$$
\begin{aligned}
&\big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\big\rangle=\\
&\qquad=\big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v),\mathbf{e}_{\vec k}\big\rangle\big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{f}^{-\vec r}\big\rangle=\big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{f}^{-\vec r}\big\rangle\big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v),\mathbf{e}_{\vec k}\big\rangle,
\end{aligned}
$$

which follows from the relation $\mathbf{b}^{(+)}_{1^*}(x)\mathbf{b}^{(-)}_2(y)=\mathbf{b}^{(-)}_2(y)\mathbf{b}^{(+)}_{1^*}(x)$. We calculate action of the operators $T^{\pm n}_{\pm n}(x)$ and $\tilde{\mathbf{T}}^{(\pm)}_0(x)$ on the element $\big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\big\rangle$ in both orders and then we get assertion of Lemma 5 by comparing these expressions.
From Lemmas 3 and 4 we get

$$
\begin{aligned}
&T^n_n(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle=T^n_n(x)\big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v),\mathbf{e}_{\vec k}\big\rangle\big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{f}^{-\vec r}\big\rangle=\\
&\quad=F(\overline{v};x)F(\overline{w};x-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle T^n_n(x)-\\
&\qquad-\sum_{v_\ell\in\overline{v}} g(v_\ell,x)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec v_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec w),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec v)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle T^n_n(v_\ell)+\\
&\qquad+\sum_{w_s\in\overline{w}}F(\overline{w}_s;w_s)\,\mathrm{Tr}_0\Big[F(\overline{v};x)\tilde h(w_s,x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec v)\mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec w_s),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec w)\widehat{\mathbb{T}}^{(-,-)}_{s;0;1^*,\dots,Q^*}(\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle-\\
&\qquad\qquad-\sum_{v_\ell\in\overline{v}} g(v_\ell,x)\tilde h(w_s,v_\ell)F(\overline{v}_\ell;v_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec v_\ell)\mathbf{b}^{(+)}_{s^*}(v_\ell)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec w_s),\\
&\qquad\qquad\qquad\qquad\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec w)\widehat{\mathbb{T}}^{(-,-)}_{s;0;1^*,\dots,Q^*}(\vec w)\mathbf{e}_{\vec k}\otimes\mathbf{f}^{-\vec r}\Big\rangle\Big]
\end{aligned}
$$

In the first term the vectors $\vec{v}$ and $\vec{w}$ do not change, in the second the components v_ℓ and x are interchanged and the third contains expressions in which w_s and x are interchanged. On the other hand, we also have

$$
\begin{aligned}
&T^n_n(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = T^n_n(x)\langle\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\rangle\langle\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\rangle =\\
&\quad = F(\overline{v};x)F(\overline{w};x-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(x)-\\
&\quad\quad -\sum_{v_\ell\in\overline{v}}F(\overline{v}_\ell;v_\ell)\Bigg(g(v_\ell,x)F(\overline{w};x-n+1+\eta)\\
&\qquad\qquad\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\qquad\quad -\sum_{w_s\in\overline{w}}\tilde{h}(w_s,v_\ell)\tilde{h}(w_s,x)F(\overline{w}_s;w_s)\\
&\qquad\qquad \mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s)\mathbf{b}^{(-)}_{\ell}(w_s),\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\\
&\qquad\qquad\quad \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(w_s;\vec{w})\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle\Bigg)T^n_n(v_\ell)+\\
&\quad\quad +\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(\overline{w}_s;w_s)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad\qquad\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(w_s;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle
\end{aligned}
$$

In this expression the vectors $\vec{v}$ and $\vec{w}$ do not change in the first term, in the second x is changed by v_ℓ and in the third x and w_s are interchanged. If we compare these two expressions, we get the first statement of Lemma 5.
We get the second relation when we compare the equalities

$$
\begin{aligned}
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle =\\
&\quad = F(x;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\quad\quad -\sum_{v_\ell\in\overline{v}}g(x,v_\ell)F(v_\ell;\overline{v}_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\\
&\qquad\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(v_\ell;\vec{v};\vec{w})\,\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\quad\quad -\sum_{w_s\in\overline{w}}F(w_s;\overline{w}_s)\Bigg(\tilde{h}(w_s,x)F(x;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad\qquad\qquad \widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(x;\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\qquad\quad -\sum_{v_\ell\in\overline{v}}g(x,v_\ell)\tilde{h}(w_s,v_\ell)F(v_\ell;\overline{v}_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(+)}_{s^*}(v_\ell)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(v_\ell;\vec{v})\,\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle\Bigg)T^{-n}_{-n}(w_s)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \tilde{\mathbf{T}}^{(+)}_0(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle =\\
&\quad = F(x;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\quad\quad -\sum_{v_\ell\in\overline{v}}g(x,v_\ell)F(v_\ell;\overline{v}_\ell)\Big\langle \mathbf{b}^{(+)}_{\ell^*;1^*,\dots,P^*}(x;\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\\
&\qquad\qquad\qquad \widehat{\mathbf{R}}^{(+,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P}(\vec{v})\,\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{v_\ell\in\overline{v}}\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)\tilde{h}(w_s,v_\ell)F(v_\ell;\overline{v}_\ell)F(w_s;\overline{w}_s)\\
&\qquad\mathrm{Tr}_{\widehat{0}}\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s)\mathbf{b}^{(-)}_{\ell}(w_s),\\
&\qquad\qquad\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;\widehat{0};1,\dots,P}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&-\sum_{w_s\in\overline{w}}\tilde{h}(w_s,x)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{s^*}(x)\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s),\\
&\qquad\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)
\end{aligned}
$$

The third equality is obtained by comparing the equalities

$$
\begin{aligned}
&T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle =\\
&\quad= F(x;\overline{w})F(x+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(x)-\\
&\qquad-\sum_{w_s\in\overline{w}}g(x,w_s)F(w_s;\overline{w}_s)F(x+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s),\\
&\qquad\qquad\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)-\\
&\qquad-\sum_{w_s\in\overline{w}}\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)\tilde{h}(w_s,v_\ell)F(w_s;\overline{w}_s)F(v_\ell;\overline{v}_\ell)\\
&\qquad\quad\mathrm{Tr}_0\bigg(\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(+)}_{s^*}(v_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,\widehat{s},\dots,Q}(\vec{w}_s),\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\\
&\qquad\qquad\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbb{R}}^{(+,-)}_{0_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{R}}^{(+,+)}_{0;1,\dots,P}(v_\ell;\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle\bigg)T^{-n}_{-n}(w_s)+\\
&\qquad+\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(v_\ell;\overline{v}_\ell)\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\\
&\qquad\qquad\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,P;1^*,\dots,Q^*}(v_\ell;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = T^{-n}_{-n}(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w})\mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle =\\
&\quad= F(x;\overline{w})F(x+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(x)-\\
&\qquad-\sum_{w_s\in\overline{w}}g(x,w_s)F(w_s;\overline{w}_s)F(w_s+n-1-\eta;\overline{v})\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s),\\
&\qquad\qquad\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^{-n}_{-n}(w_s)+\\
&\qquad+\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(v_\ell;\overline{v}_\ell)F(x;\overline{w})\,\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w})\mathbf{b}^{(-)}_{\ell}(x),\\
&\qquad\qquad\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle-\\
&\qquad-\sum_{v_\ell\in\overline{v}}\sum_{w_s\in\overline{w}}g(x,w_s)\tilde{h}(w_s,v_\ell)F(v_\ell;\overline{v}_\ell)F(w_s;\overline{w}_s)\\
&\qquad\quad\mathrm{Tr}_0\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell^*},\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s)\mathbf{b}^{(-)}_{\ell}(w_s),\\
&\qquad\qquad\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{T}}^{(+,+)}_{\ell;0;1,\dots,P}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle
\end{aligned}
$$

and the fourth relation is the result of equalities

$$\begin{aligned}
&\tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle = \\
&\quad = F(\overline{w};x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle- \\
&\qquad -\sum_{w_s\in\overline{w}} g(w_s,x)F(\overline{w}_s;w_s)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s), \\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{s;0;1^*,\dots,Q^*}(\vec{w})\widehat{\mathbf{R}}^{(-,+)}_{0;1,\dots,P}(x;\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle- \\
&\qquad -\sum_{w_s\in\overline{w}}\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)\tilde{h}(w_s,v_\ell)F(\overline{w}_s;w_s)F(\overline{v}_\ell;v_\ell) \\
&\qquad\qquad \mathrm{Tr}_{\widehat{0}}\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(+)}_{s^*}(v_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots\widehat{s},\dots,Q}(\vec{w}_s),\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+} \\
&\qquad\qquad \widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\widehat{\mathbb{P}}^{(+,-)}_{s_+,s^*_-}\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbb{T}}^{(-,-)}_{s;\widehat{0};1^*,\dots,Q^*}(\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle- \\
&\qquad -\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};v_\ell-n+1+\eta)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{\ell}(x)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}), \\
&\qquad\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(v_\ell)
\end{aligned}$$

$$\begin{aligned}
&\tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle = \tilde{\mathbf{T}}_0^{(-)}(x)\Big\langle \mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\mathbf{f}^{-\vec{r}}\Big\rangle\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v}),\mathbf{e}_{\vec{k}}\Big\rangle = \\
&\quad = F(\overline{w};x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w}),\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(x;\vec{v};\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle- \\
&\qquad -\sum_{w_s\in\overline{w}} g(w_s,x)F(\overline{w}_s;w_s)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,P^*}(\vec{v})\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s), \\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,P;1^*,\dots,Q^*}(w_s;\vec{v},\vec{w})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle- \\
&\qquad -\sum_{v_\ell\in\overline{v}}\tilde{h}(x,v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w};x)\Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{1,\dots,Q}(\vec{w})\mathbf{b}^{(-)}_{\ell}(x), \\
&\qquad\qquad \widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(x;\vec{w})\widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(v_\ell)+ \\
&\qquad +\sum_{v_\ell\in\overline{v}}\sum_{w_s\in\overline{w}} g(w_s,x)\tilde{h}(w_s,v_\ell)F(\overline{v}_\ell;v_\ell)F(\overline{w}_s;w_s) \\
&\qquad\qquad \Big\langle \mathbf{b}^{(+)}_{1^*,\dots,\widehat{\ell}^*,\dots,P^*}(\vec{v}_\ell)\mathbf{b}^{(-)}_{s;1,\dots,Q}(x;\vec{w}_s)\mathbf{b}^{(-)}_{\ell}(w_s),\widehat{\mathbf{R}}^{(-,-)}_{1^*,\dots,s^*}(\vec{w})\widehat{\mathbf{R}}^{(-,-)}_{0;1^*,\dots,Q^*}(w_s;\vec{w}) \\
&\qquad\qquad \widehat{\mathbb{P}}^{(-,+)}_{\ell^*_-,\ell_+}\widehat{\mathbb{R}}^{(-,+)}_{0_-,\ell_+}\widehat{\mathbf{R}}^{(+,+)}_{1,\dots,\ell}(\vec{v})\mathbf{e}_{\vec{k}}\otimes\mathbf{f}^{-\vec{r}}\Big\rangle T^n_n(v_\ell)
\end{aligned}$$

□